University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 2
solutions

1. A projectile that is subject to quadratic air resistance is thrown vertically *up* with initial speed v_o. **(a)** Write down the equation of motion for the upward motion and solve it to give $v(t)$. **(b)** Show that the time to reach the top of the trajectory is

$$t_{\text{top}} = \frac{v_{\text{ter}}}{g} \arctan\left(\frac{v_o}{v_{\text{ter}}}\right) \tag{1}$$

(c) For the baseball of Example 2.5 in your text (with $v_{\text{ter}} = 35\,\text{m/s}$), find t_{top} for the cases that $v_o = 1, 10, 20, 30, 40\,\text{m/s}$ and compare with the corresponding values in a vacuum.

Solution: **(a)** Starting from Newton's laws, write down the force balance. Then separate variables and integrate.

$$m\dot{v} = m\frac{dv}{dt} = -mg - cv^2 \tag{2}$$

$$\frac{dv}{dt} = -g - \frac{cv^2}{m} \tag{3}$$

$$dt = -\frac{dv}{g + cv^2/m} \tag{4}$$

$$t = -\int\limits_{v_o}^{v} \frac{dv'}{g + c(v')^2/m} = -\left.\frac{\arctan\left(v\sqrt{\frac{c}{mg}}\right)}{\sqrt{gc/m}}\right|_{v_o}^{v} \tag{5}$$

$$t\sqrt{\frac{gc}{m}} = \arctan\left(v_o\sqrt{\frac{c}{mg}}\right) - \arctan\left(v\sqrt{\frac{c}{mg}}\right) \tag{6}$$

$$\arctan\left(v\sqrt{\frac{c}{mg}}\right) = \arctan\left(v_o\sqrt{\frac{c}{mg}}\right) - t\sqrt{\frac{gc}{m}} \qquad \text{note } v_{\text{ter}} = \sqrt{\frac{mg}{c}} \tag{7}$$

$$\arctan\left(\frac{v}{v_{\text{ter}}}\right) = \arctan\left(\frac{v_o}{v_{\text{ter}}}\right) - \frac{gt}{v_{\text{ter}}} \tag{8}$$

$$v = v_{\text{ter}} \tan\left[\arctan\left(\frac{v_o}{v_{\text{ter}}}\right) - \frac{gt}{v_{\text{ter}}}\right] \tag{9}$$

(b) At the top, $v=0$, so the argument of the tan function in equation 9 above must be zero.

$$0 = \arctan\left(\frac{v_o}{v_{\text{ter}}}\right) - \frac{gt_{\text{top}}}{v_{\text{ter}}} \qquad \Longrightarrow \qquad t_{\text{top}} = \frac{v_{\text{ter}}}{g} \arctan\left(\frac{v_o}{v_{\text{ter}}}\right) \tag{10}$$

(c) The time to the top without air resistance is the familiar result $t_{\text{top}} = v_o/g$. With the numbers given and $g = 9.81\,\text{m/s}^2$:

$\boldsymbol{v_o}$ (m/s)	$\boldsymbol{t_{\text{top}}}$ (s; with drag)	t_{top} (s; no drag)
1	0.110	0.102
10	0.993	1.019
20	1.852	2.039
30	2.528	3.058
40	3.040	4.077

2. Two people, each of mass m_h, are standing at one end of a stationary railroad flatcar with frictionless wheels and mass m_{fc}. Either person can run to the other end of the flatcar and jump off with the same speed u (relative to the car). **(a)** Use conservation of momentum to find the speed of the recoiling car if the two people run and jump simultaneously. **(b)** What is it if the second person starts running only after the first has jumped? Which procedure gives the greater speed to the car? [*Hint:* The speed u is the speed of either person, *relative to the car* just after they have jumped; it has the same value for either person and is the same in parts (a) and (b).]

Solution: **(a)** Assume both people jump along $+x$. Let the flatcar's recoil velocity be v, so that $u - v$ is the speed of either person relative to the ground just after they jump. Conservation of momentum implies

$$2m_h(u - v) = m_{fc}v \tag{11}$$

$$v = \frac{2m_h}{2m_h + m_{fc}}u \tag{12}$$

(b) Let v' be the recoil speed of the flatcar just after the first person jumps and v'' that after the second person jumps. For the first jump, conservation of momentum works just like it did above except that only one person jumps:

$$m_h(u - v) = (m_{fc} + m_h)v \tag{13}$$

$$v' = \frac{m_h}{2m_h + m_{fc}}u \tag{14}$$

For the second jump, we have to account for the fact that the flatcar is already moving with speed v' along $-x$. In this case, conservation of momentum gives

$$p_i = p_f \tag{15}$$

$$-(m_h + m_{fc})v' = m_h(u - v'') - m_{fc}v'' \tag{16}$$

Simplifying,

$$v'' = \frac{m_h u + (m_h + m_{fc})v'}{m_h + m_{fc}} = \frac{2m_h(3m_h + 2m_{fc})}{(2m_h + 2m_{fc})(2m_h + m_{fc})}u \tag{17}$$

In order to compare with the result of (a), we can use Eq. 12 to relate u and v:

$$v'' = \frac{3m_h + 2m_{fc}}{2m_h + 2m_{fc}}v \tag{18}$$

Since the fraction above is always greater than 1, it is clear that $v'' > v$ so the second method gives the larger final velocity.

3. Many applications of conservation of momentum involve conservation of energy as well, and we haven't yet begun our discussion of energy. Nevertheless, you know enough about energy from your introductory physics course to handle some problems of this type. Here is one elegant example: an *elastic* collision between two bodies is defined as a collision in which the total kinetic energy of the two bodies is the same before and after the collision (for example, the collision of two billiard balls, which generally lose extremely little of their kinetic energy.) Consider an elastic collision between two equal mass bodies, one of which is initially at rest. Let their velocities be $\mathbf{v}_1$ and $\mathbf{v}_2 = 0$ before the collision, and $\mathbf{v}'_1$ and $\mathbf{v}'_2$ after. Write down the vector equation representing the conservation of momentum and the scalar equation which expresses that the collision is elastic. Use these to prove that the angle between $\mathbf{v}'_1$ and $\mathbf{v}'_2$ is 90°. This result was important in the history of atomic and nuclear physics: that two bodies emerged from a collision traveling on perpendicular paths was strongly suggestive that they had equal mass and had undergone an elastic collision.

Solution: Momentum tells us (since the masses are equal)

$$\mathbf{v}'_1 + \mathbf{v}'_2 = \mathbf{v}_1 = v_1\hat{\mathbf{x}} \tag{19}$$

where we have chosen the x axis to be the direction of $\mathbf{v}_1$ for simplicity. Square that to find the magnitude of both sides:

$$v_1^2 = |v'_1 + v'_2|^2 = (v'_1)^2 + (v'_2)^2 - 2v'_1v'_2\cos\theta \tag{20}$$

Here θ is the angle between $\mathbf{v}'_1$ and $\mathbf{v}'_2$. Since the masses are the same, conservation of kinetic energy adds:

$$v_1^2 = (v'_1)^2 + (v'_2)^2 \tag{21}$$

Clearly, the last two equations can only both be true, aside from the trivial case of $v'_1 = v'_2 = 0$, if

$\cos\theta = 0$, or $\theta = 90°$.

4. A rocket (initial mass m_o) needs to use its engines to hover stationary, just above the ground. **(a)** If it an afford to burn no more than a mass λm_o of its fuel, how long can it hover? [*Hint:* Write down the condition that the thrust just balances the force of gravity. You can integrate the resulting equation by separating the variables t and m. Take v_{ex} to be constant.] **(b)** If $v_{ex} \approx 3000\,\mathrm{m/s}$ and $\lambda \approx 10\%$, for how long could the rocket hover just above the earth's surface?

Solution: **(a)** The rocket's thrust is $-\dot{m}v_{\text{ex}}$, which should just balance the force of gravity, mg, noting that m is a function of time. Thus

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg = 0 \tag{22}$$

$$\implies \quad \dot{m}v_{\text{ex}} = -mg \tag{23}$$

$$\tag{24}$$

The negative sign makes sense, since gravity opposes the thrust. Now we separate variables and integrate. We start with mass m_o and end with mass $m_o - \lambda m_o$.

$$\dot{m}v_{\text{ex}} = \frac{dm}{dt}v_{\text{ex}} = -mg \tag{25}$$

$$dt = -\frac{v_{\text{ex}}}{g}\frac{dm}{m} \tag{26}$$

Now integrate from initial to final mass:

$$t = -\int_{m_o}^{m_o-\lambda m_o} \frac{v_{\text{ex}}}{g}\frac{dm}{m} = -\frac{v_{\text{ex}}}{g}\left(\ln m\right)\Big|_{m_o}^{m_o-\lambda m_o} = -\frac{v_{\text{ex}}}{g}\ln\left(\frac{m_o - \lambda m_o}{m_o}\right) \tag{27}$$

$$t = -\frac{v_{\text{ex}}}{g}\ln\left(1-\lambda\right) = \frac{v_{\text{ex}}}{g}\ln\left(\frac{1}{1-\lambda}\right) \tag{28}$$

(b) With the numbers given, $t \approx 32\,\mathrm{s}$.

5. Consider a rocket (initial mass m_o) accelerating from rest in free space. At first, as it speeds up, its momentum p in creases, but as its mass m decreases p eventually begins to decrease. For what value of m (in terms of m_o) is p maximum?

Solution: We already know equation (3.8) from your textbook:

$$v - v_o = v_{\text{ex}}\ln\left(\frac{m_o}{m}\right) \tag{29}$$

With $p = mv$, starting from rest ($v_o\!=\!0$):

$$p = mv_{\text{ex}} \ln\left(\frac{m_o}{m}\right) \tag{30}$$

We can find the extreme values of p by finding dp/dm:

$$\frac{dp}{dm} = v_{\text{ex}} \ln\left(\frac{m_o}{m}\right) - mv_{\text{ex}} \cdot \frac{1}{m} = v_{\text{ex}} \left[\ln\left(\frac{m_o}{m}\right) - 1\right] = 0 \tag{31}$$

$$\implies \qquad \ln\left(\frac{m_o}{m}\right) = 1 \qquad \implies \qquad \frac{m_o}{m} = e \qquad \text{or} \qquad m = \frac{m_o}{e} \tag{32}$$

Here e is the base of the natural logarithms, 2.718... We found an extreme point, but is it a maximum? A plot of $p(m)$ makes it obvious, but we can check the second derivative:

$$\frac{d^2p}{dm^2} = -v_{\text{ex}} \cdot \frac{1}{m} < 0 \tag{33}$$

The second derivative is zero for all m, so we have found a maximum.

6. (a) Consider a rocket traveling in a straight line subject to an external force F^{ext} acting along the same line. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}} \tag{34}$$

[Review the derivation of equation (3.6) in the book but keep the external force term.] **(b)** Specialize to the case of a rocket taking off vertically (from rest) in a (constant) gravitational field g, so the equation of motion becomes

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg \tag{35}$$

Assume that the rocket ejects mass at a constant rate, $\dot{m} = -k$ (where k is a positive constant), so that $m = m_o - kt$. Solve the equation for v as a function of t. **(c)** Using the rough data from problem 3.7 in your textbook, find the space shuttle's speed two minutes into flight, assuming (what is nearly true) that is travels vertically up during this period and that g does not change appreciably. Compare with the corresponding result if there were no gravity. **(d)** Describe what would happen to a rocket that was designed so that the first term on the right of equation 35 was smaller than the initial value of the second.

Solution: **(a)** Start with momentum conservation. Aside from the rocket's change in momentum, we have to add in the impulse from the external force. In analogy with equation 3.4 in your text, we can write down the rocket's change in momentum.

$$dP_{\text{tot}} = P(t+dt) - P(t) = m\,dv + dm\,v_{\text{ex}} \tag{36}$$

The rocket's change in momentum should equal the impulse delivered by the external force:

$$F^{\text{ext}}\,dt = m\,dv + dm\,v_{\text{ex}} \tag{37}$$

$$F^{\text{ext}} = m\dot{v} + \dot{m}v_{\text{ex}} \tag{38}$$

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}} \tag{39}$$

(b) With $F^{\text{ext}} = -mg$, and $\dot{m} = kt$ such that $m = m_o - kt$,

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg \tag{40}$$

$$\dot{v} = \frac{dv}{dt} = -\frac{\dot{m}}{m}v_{\text{ex}} - g = -\frac{dm}{dt}\frac{v_{\text{ex}}}{m} - g \tag{41}$$

$$dv = -\frac{dm}{m}v_{\text{ex}} - g\,dt \tag{42}$$

Now we can integrate both sides

$$\int_0^v dv' = \int_{m_o}^m -\frac{dm'}{m'}v_{\text{ex}} - \int_0^t g\,dt' \tag{43}$$

$$v = -v_{\text{ex}}\left(\ln m - \ln m_o\right) - gt \tag{44}$$

$$v = -v_{\text{ex}}\ln\left(\frac{m}{m_o}\right) - gt = v_{\text{ex}}\ln\left(\frac{m_o}{m_o - kt}\right) - gt \tag{45}$$

(c) With $v_{\text{ex}} \approx 3000\,\text{m/s}$, $m_o/m \approx 2$, $t = 120\,\text{s}$, we find $v \approx 900\,\text{m/s}$ compared to $\approx 2100\,\text{m/s}$ without gravity. **(d)** If $-\dot{m}v_{\text{ex}} < -mg$, the rocket's thrust is less than the force of gravity and the rocket cannot leave the ground until it sheds enough mass that the thrust can overcome the rocket's weight. Not a good design.

7. Use the results of the previous problem giving $v(t)$ for a rocket accelerating vertically from rest in a gravitational field g. Now integrate $v(t)$ to show the rocket's height as a function of t is

$$y(t) = v_{\text{ex}}t - \frac{1}{2}gt^2 - \frac{mv_{\text{ex}}}{k}\ln\left(\frac{m_o}{m}\right) \tag{46}$$

Using the numbers given in problem 3.7 in your textbook, estimate the space shuttle's height after 2 minutes.

Solution: We just have to integrate the result of the previous problem.

$$y = \int_0^t v\,dt = \int_0^t v_{\text{ex}} \ln\left(\frac{m_o}{m_o - kt}\right) - gt\,dt = -v_{\text{ex}}\left(t - \frac{m_o}{k}\right)\ln\left(\frac{m_o - kt}{m_o}\right) + v_{\text{ex}}t - \frac{1}{2}gt^2\Big|_0^t$$

$$y = -v_{\text{ex}}\left(t - \frac{m_o}{k}\right)\ln\left(\frac{m_o - kt}{m_o}\right) - \frac{m_o v_{\text{ex}}}{k}\ln 1 + v_{\text{ex}}t - \frac{1}{2}gt^2 \tag{47}$$

$$y = v_{\text{ex}}\left(\frac{m_o}{k} - t\right)\ln\frac{m}{m_o} + v_{\text{ex}}t - \frac{1}{2}gt^2 \tag{48}$$

$$y = v_{\text{ex}}t - \frac{1}{2}gt^2 - \frac{m v_{\text{ex}}}{k}\ln\frac{m_o}{m} \tag{49}$$

With $v_{\text{ex}} \approx 3000\,\text{m/s}$, $t = 120\,\text{s}$, $m_o/m = 2$, and $m = 1\times 10^6\,\text{kg}$ we can first find k:

$$m = m_o - kt \tag{50}$$

$$1\times 10^6\,\text{kg} = 2\times 10^6\,\text{kg} - k(120\,\text{s}) \tag{51}$$

$$k = 8333\,\text{kg/s} \tag{52}$$

With these numbers, we find $y \approx 4\times 10^4\,\text{m}$.

8. **(a)** We know that the path of a projectile thrown from the ground is a parabola if we ignore air resistance. In the light of equation (3.12) in your textbook ($\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$), what would be the subsequent path of the CM of the pieces if the projectile exploded in midair? **(b)** A shell is fired from ground level so as to hit a target 100 m away. Unluckily, the shell explodes prematurely and breaks into two equal pieces. The two pieces land at the same time, and one lands 100 m beyond the target. Where does the other piece land? **(c)** Is the same result true if they land at different times (with one piece still landing 100 m beyond the target)?

Solution: **(a)** Only internal forces act, so from momentum conservation the CM must continue along a parabolic path. **(b)** For the CM trajectory to be unchanged, the other piece must land 100 m short of the target, meaning it will hit the gun that fired it! **(c)** If one piece lands 100 m beyond the target and the second piece lands at a different time, the CM trajectory is no longer the original parabolic path, and there must be an external force F acting on the projectile to break into two equal pieces. Originally we had $\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}} = M\mathbf{g}$, now we have $\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}} = M\mathbf{g} + \mathbf{F} \neq Mg$

9. Use spherical polar coordinates (r, θ, φ) to find the CM of a uniform solid hemisphere of radius R whose flat face lies in the xy plane with its center at the origin. You will need the element of volume in spherical polar coordinates.

Solution: By symmetry, we know $\overline{x} = \overline{y} = 0$. That means we need only find the CM position for the z axis. The distance to the z axis is $r\cos\varphi$. Assuming a constant density σ and volume element $r^2 \sin\varphi\, dr\, d\theta\, d\varphi$,

$$\overline{z} = \frac{\int z\,dm}{\int dm} = \frac{\int_0^{\pi/2}\int_0^{2\pi}\int_0^R (r\cos\varphi)\,\sigma r^2 \sin\varphi\,dr\,d\theta\,d\varphi}{\int_0^{\pi/2}\int_0^{2\pi}\int_0^R \sigma r^2 \sin\varphi\,dr\,d\theta\,d\varphi} = \frac{\frac{1}{4}R^4\sigma \int_0^{\pi/2}\int_0^{2\pi} \cos\varphi\sin\varphi\,d\theta\,d\varphi}{\frac{1}{3}R^3\sigma \int_0^{\pi/2}\int_0^{2\pi} \sin\varphi\,d\varphi\,d\theta} \tag{53}$$

$$= \frac{3R}{4}\frac{2\pi\int_0^{\pi/2} \cos\varphi\sin\varphi\,d\varphi}{2\pi\int_0^{\pi/2} \sin\varphi\,d\varphi} = \frac{3R}{4}\frac{\frac{1}{2}}{1} = \frac{3R}{8} \tag{54}$$

10. I have a hemispherical bowl of radius R, and I wish to fill it to a height h such that half the volume is filled. To what height h (in terms of R) do I need to fill it? (Imagine you have a hemispherical 1 tsp measuring spoon and need to fill it to 1/2 tsp.)

Solution: Assume the flat face of the bowl lies in the $x-y$ plane at $z=0$, with the bottom at $z=-R$. If the bowl is filled to height z, at that height $(R-z)^2+r^2=R^2$, where r is the radius at height z. That means $r^2=2Rz-z^2$. The volume of a circular segment of width dz at height z is then $\pi r^2\,dz$. The volume when filling to a height h is then

$$V = \int_0^h \pi r^2\,dz = \int_0^h \pi(2Rz-z^2)\,dz = \pi Rz^2 - \frac{\pi}{3}z^3\Big|_0^h = \pi Rh^2 - \frac{\pi h^3}{3} = \frac{1}{3}\pi h^2(3R-h) \tag{55}$$

At $h=R$ the volume is $2\pi R^3/3$ as expected. To fill the bowl to half volume, we need $V=\pi R^3/3$. This gives us

$$h^2(3R-h) = R^3 \tag{56}$$

There is in fact an analytic solution involving various roots of i, but it is not terribly enlightening. A numerical solution gives $h \in \{-0.53209, 0.65270, 2.8794\}R$. Only $h=0.65270R$ falls within the valid range of heights. So, you can fill your bowl about 65% of the way up to use half the volume. Or, to better than 2%, just fill it to 2/3 the height and you'll have half the volume.