UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 3

due 14 Sept 2018

Instructions:

1. Answer all questions below (bonus questions optional).
2. Show your work for full credit.
3. All problems are due by 11:59pm on 14 Sept 2018.
4. You may collaborate, but everyone must turn in their own work.

1. A particle moves under the influence of a central force directed toward a fixed origin O. **(a)** Explain why the particle's angular momentum about O is constant. **(b)** Give in detail the argument that the particle's orbit must lie in a single plane containing O.

2. Consider a planet orbiting a fixed sun. Take the plane of the planet's orbit to be the xy plane, with the sun at the origin, and label the planet's position by polar coordinates (r, φ). **(a)** Show that the planet's angular momentum has magnitude $l = mr^2\omega$, where $\omega = \dot{\varphi}$ is the planet's angular velocity about the sun. **(b)** Show that the rate at which the planet "sweeps out area" (as in Kepler's second law) is $dA/dt = \frac{1}{2}r^2\omega$, and hence $dA/dt = l/2m$. Deduce Kepler's second law.

3. A juggler is juggling a uniform rod one end of which is coated in tar and burning. He is holding the rod by the opposite end and throws it up so that, at the moment of release, it is horizontal, its CM is traveling vertically up at speed v_o and it is rotating with angular velocity ω_o. To catch it, he wants to arrange that when it returns to his hand it will have made an integer number of complete rotations. What should v_o be, if the rod is to have exactly n rotations when it returns to his hand?

4. A system consists of N masses m_α at positions $\mathbf{r}_\alpha$ relative to a fixed origin O. Let $\mathbf{r}'_\alpha$ denote the position of m_α relative to the CM; that is, $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$. **(a)** Make a sketch to illustrate this last equation. **(b)** Prove the useful relations that $\sum m_\alpha \mathbf{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious? **(c)** Use this relation to prove the result (3.28) in your textbook that the rate of change of the angular momentum *about the CM* is equal to the total external torque about the CM. (This result is surprising since the CM may be accelerating, so that it is not necessarily a fixed point in any inertial frame.)

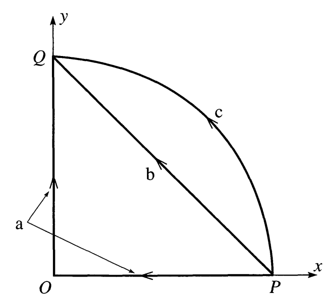
5. An infinitely long, uniform rod of mass μ per unit length is situated on the z axis. **(a)** Calculate the gravitational force $\mathbf{F}$ on a point mass m a distance ρ from the z axis. **(b)** Rewrite $\mathbf{F}$ in terms of the rectangular coordinates (x, y, z) of the point and verify that $\nabla \times \mathbf{F} = 0$. **(c)** Show that

$\nabla \times \mathbf{F} = 0$ using the expression for $\nabla \times \mathbf{F}$ in cylindrical polar coordinates (given inside the back cover of your textbook). **(d)** Find the corresponding potential energy U.

6. Evaluate the work done

$$W = \int_O^P \mathbf{F} \cdot d\mathbf{r} = \int_O^P (F_x\,dx + F_y\,dy) \tag{1}$$

by the two dimensional force $\mathbf{F} = (-y, x)$ for the three paths joining P and Q show in the figure below and defined as follows: **(a)** This path goes straight form $P = (1,0)$ to the origin and straight to $Q = (0,1)$. **(b)** This is a straight line from P to Q. (Write y as a function of x and rewrite the integral as an integral over x. **(c)** This is a quarter-circle centered on the origin. (Write x and y in polar coordinates and rewrite the integral as an integral over φ.)



(b)

7. A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius r_o with angular velocity ω_o, but now I pull the string down through the hole until a length r remains between the hole and the particle. **(a)** What is the particle's angular velocity now? **(b)** Assuming that I pull the string so slowly that we can approximate the particle's path by a circle of slowly shrinking radius, calculate the work I did in pulling the string. **(c)** Compare your answer to part (b) with the particle's gain in kinetic energy.

8. Consider a small frictionless puck perched at the top of a fixed sphere of radius R. If the puck is given a tiny nudge so that it begins to slide down, through what vertical height will it descend before it leaves the sphere? [*Hint:* Use conservation of energy to find the puck's speed as a function of its height, then use Newton's second law to find the normal force of the sphere on the puck. At what value of this normal force does the puck leave the sphere?]

9. A mass m is in a uniform gravitational field, which exerts the usual force $F = mg$ vertically down, but with g varying according to time, $g = g(t)$. Choosing axes with y measured vertically up and defining $U = mgy$ as usual, show that $\mathbf{F} = -\nabla U$ as usual, but, by differentiating $E = \frac{1}{2}mv^2 + U$ with respect to t, show that E is not conserved.

10. Consider a mass m on the end of a spring of force constant k and constrained to move along the horizontal x axis. If we place the origin at the spring's equilibrium position, the potential energy is $\frac{1}{2}kx^2$. At time $t = 0$ the mass is sitting at the origin and is given a sudden kick to the right so that it moves out to a maximum displacement at $x_{\text{max}} = A$ and then continues to oscillate about the origin. **(a)** Write down the equation for conservation of energy and solve it to give the mass's velocity $\dot{x}$ in terms of the position x and the total energy E. **(b)** Show that $E = \frac{1}{2}kA^2$, and use this to eliminate E from your expression for $\dot{x}$. Use this result in $t = \int dx'/\dot{x}(x')$ (4.58 in your textbook), to find the time for the mass to move from the origin out to a position x. **(c)** Solve the result of part (b) to give x as a function of t and show that the mass executes simple harmonic motion with period $2\pi\sqrt{m/k}$.