UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 3
solution

1. A particle moves under the influence of a central force directed toward a fixed origin O. **(a)** Explain why the particle's angular momentum about O is constant. **(b)** Give in detail the argument that the particle's orbit must lie in a single plane containing O.

Solution: **(a)** Let the force be $\mathbf{F} = f_r\,\hat{\mathbf{r}}$.

$$\dot{\mathbf{l}} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (f_r\hat{\mathbf{r}}) = (r\hat{\mathbf{r}}) \times (f_r\hat{\mathbf{r}}) = rf_r(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = 0 \tag{1}$$

The fact that $\dot{\mathbf{l}} = 0$ indicates $\mathbf{l}$ is constant. **(b)** $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ means $\mathbf{l}$ is perpendicular to a plane formed by $\mathbf{r}$ and $\mathbf{p}$. That means $\mathbf{p}$ lies in a plane with $\mathbf{r}$, and because $\mathbf{r}$ points toward O, this plane must contain O in order to keep $\mathbf{l}$ fixed.

2. Consider a planet orbiting a fixed sun. Take the plane of the planet's orbit to be the xy plane, with the sun at the origin, and label the planet's position by polar coordinates (r, φ). **(a)** Show that the planet's angular momentum has magnitude $l = mr^2\omega$, where $\omega = \dot{\varphi}$ is the planet's angular velocity about the sun. **(b)** Show that the rate at which the planet "sweeps out area" (as in Kepler's second law) is $dA/dt = \frac{1}{2}r^2\omega$, and hence $dA/dt = l/2m$. Deduce Kepler's second law.

Solution: **(a)**Because we have a central force, $\mathbf{v}$ is tangential and perpendicular to $\mathbf{r}$. This means $|\mathbf{v}| = r\dot{\varphi} = r\omega$. From the definition of $\mathbf{l}$:

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v}) \tag{2}$$

$$|\mathbf{l}| = m(r)(r\omega) = mr^2\omega \tag{3}$$

(b) Over a slice of angle $d\varphi$, with constant radius r, the area is $dA = \frac{1}{2}(r)(r\,d\varphi) = \frac{1}{2}r^2\,d\varphi$, as shown in Fig. 1 (for infinitesimal $d\varphi$, we treat the pie slice as a triangle). Since r is constant, $\dot{r} = 0$, and the rate of change of dA is determined only by φ,

$$dA = \frac{1}{2}r^2\,d\varphi \tag{4}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\varphi}{dt} = \frac{1}{2}r^2\dot{\varphi} = \frac{1}{2}r^2\omega \tag{5}$$

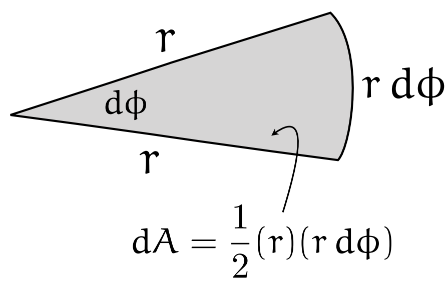
Or, starting from equation (3.24) in your text, with $\dot{r}$ =constant,

$$dA = \frac{1}{2}|\mathbf{r} \times r\, d\boldsymbol{\varphi}| \tag{6}$$

$$\left|\frac{dA}{dt}\right| = \frac{1}{2}\left|\mathbf{r} \times r\frac{d\boldsymbol{\varphi}}{dt}\right| = \frac{1}{2}r^2\omega \qquad \text{since } \mathbf{r} \perp \boldsymbol{\varphi} \tag{7}$$

$$\left|\frac{dA}{dt}\right| = \frac{|\mathbf{l}|}{2m} \tag{8}$$

Figure 1: *Area of a segment $d\varphi$ of a circle of radius r.*



3. A juggler is juggling a uniform rod one end of which is coated in tar and burning. He is holding the rod by the opposite end and throws it up so that, at the moment of release, it is horizontal, its CM is traveling vertically up at speed v_o and it is rotating with angular velocity ω_o. To catch it, he wants to arrange that when it returns to his hand it will have made an integer number of complete rotations. What should v_o be, if the rod is to have exactly n rotations when it returns to his hand?

Solution: We need the time of flight to be equal to an integer number of rotation periods. This also means the net angular displacement should be a multiple of 2π, so $\Delta\theta = \omega_o t = 2\pi n$. The time of flight starting and ending at the origin (setting the origin at the launch point) is given by

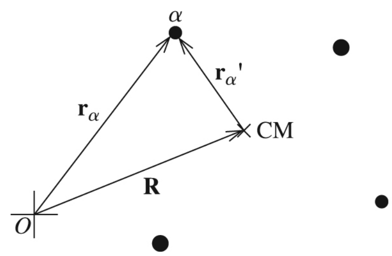
$$y = v_o t - \frac{1}{2}gt^2 = 0 \qquad \Longrightarrow \qquad t = \frac{2v_o}{g} \tag{9}$$

Equating that to the rotation time will satisfy the condition that the rod arrives back at the origin after an integer number of full rotations.

$$\omega_o t = \omega_o\left(\frac{2v_o}{g}\right) = 2\pi n \qquad \Longrightarrow \qquad v_o = \frac{n\pi g}{\omega_o} \tag{10}$$

4. A system consists of N masses m_α at positions $\mathbf{r}_\alpha$ relative to a fixed origin O. Let $\mathbf{r}'_\alpha$ denote the position of m_α relative to the CM; that is, $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$. **(a)** Make a sketch to illustrate this last equation. **(b)** Prove the useful relations that $\sum m_\alpha \mathbf{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious? **(c)** Use this relation to prove the result (3.28) in your textbook that the rate of change of the angular momentum *about the CM* is equal to the total external torque about the CM. (This result is surprising since the CM may be accelerating, so that it is not necessarily a fixed point in any inertial frame.)

Solution: **(a)** Noting $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$,



Problem 3.37
Taylor CLASSICAL MECHANICS

(b) In the $x'-y'$ system, the CM is at $\mathbf{R}' = 0$. Writing down the sum and substituting the definition of $\mathbf{r}'_\alpha$,

$$\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha \left(\mathbf{r}_\alpha - \mathbf{R}\right) = \sum m_\alpha \mathbf{r}_\alpha - \left(\sum m_\alpha\right)\mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0 \tag{11}$$

This is equivalent to finding the net torque due to gravity about the CM, which is zero since the gravitational force acts on the center of mass. More to the point, the sum $(1/M)\sum m_\alpha \mathbf{r}'_\alpha$ defines the position of the CM relative to the CM, which is clearly zero.

(c) The angular momentum about the CM is

$$\mathbf{L}(\text{about CM}) = \sum \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha \tag{12}$$

Taking the time derivative,

$$\begin{aligned}
\dot{\mathbf{L}} &= \sum \dot{\mathbf{r}}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha + \sum \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}'_\alpha && (13)\\
&= 0 + \sum \mathbf{r}'_\alpha \times m_\alpha \left(\ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}}\right) && (14)\\
&= \sum \mathbf{r}'_\alpha \times \mathbf{F}_\alpha - \left(\sum m_\alpha \mathbf{r}'_\alpha\right) \times \ddot{\mathbf{R}} && (15)\\
&= \mathbf{\Gamma}(\text{about CM}) - 0 && (16)\\
&= \mathbf{\Gamma}^{\text{ext}}(\text{about CM}) && (17)
\end{aligned}$$

The first sum on the right in the first line is zero because the cross product of two parallel vectors (in this case the cross product of the same vector) is zero. The second sum on the third line is zero based on the result of part (b). Finally, in the last line we can say $\mathbf{\Gamma} = \mathbf{\Gamma}^{\text{ext}}$ since we have already established that the sum of internal torques cancels.

(Alternate, longer solution to part c): First, establish that $\mathbf{L}_o = \mathbf{L}_{\text{o,cm,orbit}} + \mathbf{L}_{\text{o,cm,spin}}$, where "spin" means rotation of the body about its center of mass and "orbit" means the entire body rotating around the origin. Also note $\mathbf{r}_\alpha = \mathbf{r}'_\alpha + \mathbf{R}$ (with the same relationship holding after a time derivative on either side):

$$\mathbf{L}_{o_\alpha} = \mathbf{r}_\alpha \times \mathbf{p}_\alpha = \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha \qquad \text{for any } m_\alpha \tag{18}$$

$$\mathbf{L}_o = \sum_\alpha \mathbf{L}_{o_\alpha} = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha = \sum_\alpha m_\alpha \left(\mathbf{r}'_\alpha + \mathbf{R}\right) \times m_\alpha \left(\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}}\right) \tag{19}$$

$$= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha + m_\alpha \dot{\mathbf{R}} \times \mathbf{r}'_\alpha + m_\alpha \mathbf{R} \times \dot{\mathbf{r}}'_\alpha + m_\alpha \mathbf{R} \times \dot{\mathbf{R}} \tag{20}$$

Now recall $\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha \dot{\mathbf{r}}'_\alpha = 0$ (the second term follows from the fact that the net momentum of all m_α about the CM is zero). That means the second and third terms in equation 20 are zero.

$$\mathbf{L}_o = \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha + m_\alpha \mathbf{R} \times \dot{\mathbf{R}} \tag{21}$$

The first term is the angular momentum about the center of mass (spin), since it has the form of angular momentum expressed in terms of coordinates relative to the center of mass, and the second is the angular momentum of the CM about O (orbit), since it has the form of angular momentum expressed in terms of coordinates about the origin. Thus,

$$\mathbf{L}_o = \mathbf{L}_{\text{o,cm,spin}} + \mathbf{L}_{\text{o,cm,orbit}} \tag{22}$$

Now we wish to find the rate of change of the angular momentum.

$$\dot{\mathbf{L}}_o = \frac{d}{dt}\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha + m_\alpha \mathbf{R} \times \dot{\mathbf{R}}\right) \tag{23}$$

$$= \sum_\alpha m_\alpha \dot{\mathbf{r}}'_\alpha \times \dot{\mathbf{r}}'_\alpha + m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha + m_\alpha \dot{\mathbf{R}} \times \dot{\mathbf{R}} + m_\alpha \mathbf{R} \times \ddot{\mathbf{R}} \tag{24}$$

The first and third terms are zero since for any vector $\mathbf{a}$, $\mathbf{a} \times \mathbf{a} = 0$

$$\dot{\mathbf{L}}_o = \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha + m_\alpha \mathbf{R} \times \ddot{\mathbf{R}} = \left(\sum_\alpha \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}'_\alpha \right) + \left(\mathbf{R} \times \ddot{\mathbf{R}} \right) \sum_\alpha m_\alpha \tag{25}$$

$$= \left(\sum_\alpha \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}'_\alpha \right) + M \left(\mathbf{R} \times \ddot{\mathbf{R}} \right) = \left(\sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}} \right) + \left(\mathbf{R} \times \mathbf{F}^{\text{ext}} \right) \tag{26}$$

For the second to last line we noted that $\mathbf{R}$ and $\ddot{\mathbf{R}}$ are the same for all m_α; for the last line we noted that $\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$. The first term is $\dot{\mathbf{L}}_{\text{o,cm,spin}}$, which we want to relate to the external torque, and the second is $\dot{\mathbf{L}}_{\text{o,cm,orbit}}$. We know that $\dot{\mathbf{L}}_o = \boldsymbol{\Gamma}^{\text{ext}}$, and $\dot{\mathbf{L}}_o = \dot{\mathbf{L}}_{\text{o,cm,spin}} + \dot{\mathbf{L}}_{\text{o,cm,orbit}}$, so it follows that $\dot{\mathbf{L}}_{\text{o,cm,spin}} = \dot{\mathbf{L}}_o - \dot{\mathbf{L}}_{\text{o,cm,orbit}}$. Starting from the definition of external torque,

$$\dot{\mathbf{L}}_o = \boldsymbol{\Gamma}^{\text{ext}} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}^{\text{ext}}_\alpha = \sum_\alpha \left(\mathbf{r}'_\alpha + \mathbf{R} \right) \times \mathbf{F}^{\text{ext}}_\alpha = \left(\sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}} \right) + \left(\mathbf{R} \times \mathbf{F}^{\text{ext}} \right) \tag{27}$$

$$\dot{\mathbf{L}}_o = \left(\sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}} \right) + \dot{\mathbf{L}}_{\text{o,cm,orbit}} \tag{28}$$

The last term in the second line, we noted that $\mathbf{R} \times \mathbf{F}^{\text{ext}}$ is the torque about the origin $\dot{\mathbf{L}}_{\text{o,cm,orbit}}$. Comparing, the term in brackets is precisely what we found previously for the spin term. This will complete our proof, since $\sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}}$ is the net external torque *measured relative to the CM* (since we are using r').

$$\dot{\mathbf{L}}_{\text{o,cm,spin}} = \dot{\mathbf{L}}_o - \dot{\mathbf{L}}_{\text{o,cm,orbit}} = \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}} = \boldsymbol{\Gamma}^{\text{ext}}(\text{about CM}) \tag{29}$$

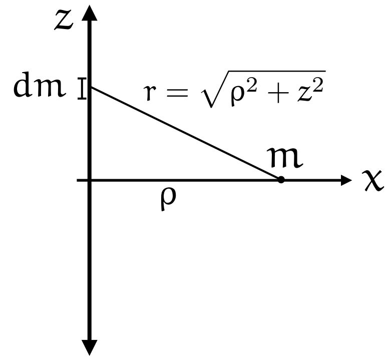
$$\text{or} \qquad \dot{\mathbf{L}}(\text{about CM}) = \boldsymbol{\Gamma}^{\text{ext}}(\text{about CM}) \tag{30}$$

5. An infinitely long, uniform rod of mass μ per unit length is situated on the z axis. **(a)** Calculate the gravitational force $\mathbf{F}$ on a point mass m a distance ρ from the z axis. **(b)** Rewrite $\mathbf{F}$ in terms of the rectangular coordinates (x, y, z) of the point and verify that $\boldsymbol{\nabla} \times \mathbf{F} = 0$. **(c)** Show that $\boldsymbol{\nabla} \times \mathbf{F} = 0$ using the expression for $\boldsymbol{\nabla} \times \mathbf{F}$ in cylindrical polar coordinates (given inside the back cover of your textbook). **(d)** Find the corresponding potential energy U.

Solution: **(a)** You may remember this problem from your introductory physics course. Here is the setup:
The force on m_1 due to an element dm as shown is

$$d\mathbf{F} = -\frac{G m_1 dm\, \hat{\mathbf{r}}}{r^2} = -\frac{G m_1 dm\, \hat{\mathbf{r}}}{\rho^2 + z^2} \tag{31}$$

Clearly only the horizontal $\hat{\boldsymbol{\rho}}$ components of the force (along the ρ direction away from z) count when we sum over all dm, by symmetry the vertical components will cancel. By symmetry we can



also just integrate z from 0 to ∞ and double the result, since the $-\infty$ to 0 segment will have the same contribution. Thus, noting $\cos\theta = \rho/\sqrt{\rho^2+z^2}$, $d\mathbf{F} = dF\cos\theta\,\hat{\boldsymbol{\rho}}$, and $dm = \mu\,dz$

$$\mathbf{F} = \int d\mathbf{F} = 2\int dF\cos\theta\,\hat{\boldsymbol{\rho}} = \int_0^\infty -\frac{2Gm_1dm}{\rho^2+z^2}\frac{\rho}{\sqrt{\rho^2+z^2}}\,\hat{\boldsymbol{\rho}} = -2Gm_1\mu\rho\int_0^\infty \frac{dz\,\hat{\boldsymbol{\rho}}}{(\rho^2+z^2)^{3/2}} \tag{32}$$

$$= -(2Gm_1\mu\rho)\left(\frac{z\,\hat{\boldsymbol{\rho}}}{\rho^2\,(\rho^2+z^2)^{1/2}}\right)\Bigg|_0^\infty = -(2Gm_1\mu\rho)\left(\frac{\hat{\boldsymbol{\rho}}}{\rho^2\,(1+\rho^2/z^2)^{3/2}}\right)\Bigg|_0^\infty \tag{33}$$

$$= -2Gm_1\mu\rho\left(\frac{1-0}{\rho^2}\right)\hat{\boldsymbol{\rho}} = -\frac{2Gm_1\mu}{\rho}\hat{\boldsymbol{\rho}} \tag{34}$$

(b) Given that we have only an $\hat{\boldsymbol{\rho}}$ component of force, and it only depends on ρ which is constant, the curl is zero by inspection. With $\rho = \sqrt{x^2+y^2}$, we can write $\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi$, with $\tan\varphi = y/x$, so the expression for force becomes

$$\mathbf{F} = -\frac{2Gm_1\mu}{\rho^2}\left(\hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,y + \hat{\mathbf{z}}\,0\right) \tag{35}$$

Since F_x depends only on x, F_y depends only on y, and $F_z = 0$, the curl is zero.

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_Z \end{vmatrix}$$

$$\boldsymbol{\nabla}\times\mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \tag{36}$$

(c) In cylindrical polars, we have $F_\varphi = F_z = 0$. What was the x axis is now the ρ axis, and thus only nonzero component is

$$\mathbf{F} = F_\rho\,\hat{\boldsymbol{\rho}} = -\frac{2Gm_1\mu}{\rho}\,\hat{\boldsymbol{\rho}} \tag{37}$$

Inspecting the curl in cylindrical polars, it is easy enough to see that if F_ρ is independent of φ and z and $F_\varphi = F_z = 0$, then $\boldsymbol{\nabla} \times \mathbf{F} = 0$.

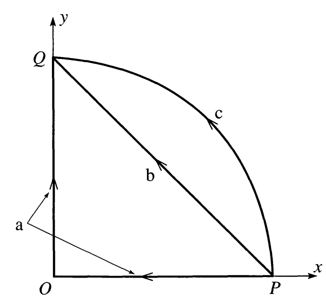
(d) We only need to integrate. Note that $U(\rho_o) = 0$.

$$U = -\int\limits_{\rho_o}^{\rho} \left(-\frac{2Gm_1\mu}{\rho'} \right) d\rho' = 2Gm_1\mu \ln\left(\frac{\rho}{\rho_o}\right) \tag{38}$$

6. Evaluate the work done

$$W = \int\limits_{O}^{P} \mathbf{F} \cdot d\mathbf{r} = \int\limits_{O}^{P} (F_x\,dx + F_y\,dy) \tag{39}$$

by the two dimensional force $\mathbf{F} = (-y, x)$ for the three paths joining P and Q show in the figure below and defined as follows: **(a)** This path goes straight form $P = (1, 0)$ to the origin and straight to $Q = (0, 1)$. **(b)** This is a straight line from P to Q. (Write y as a function of x and rewrite the integral as an integral over x. **(c)** This is a quarter-circle centered on the origin. (Write x and y in polar coordinates and rewrite the integral as an integral over φ.)



(b)

Solution: **(a)** Going from $(1, 0) \to (0, 0) \to (0, 1)$,

$$W = \int F_x\,dx + F_y\,dy = \int\limits_{x=1}^{0} -y\,dx + \int\limits_{y=0}^{1} x\,dy = 0 + 0 = 0 \tag{40}$$

Since the force is always perpendicular to the displacement, the work must be zero.

(b) Now $y = -x + 1$, and thus $dy = -dx$.

$$W = \int F_x\,dx + F_y\,dy = \int_{(1,0)}^{(0,1)} -y\,dx + x\,dy = \int_{x=1}^{0} (x-1)\,dx - x\,dx = \frac{1}{2}x^2 - x - \frac{1}{2}x^2\bigg|_0^1 = 1 \quad (41)$$

(c) Following the circle, we know

$$x = r\cos\theta \quad (42)$$
$$y = r\sin\theta \quad (43)$$
$$\mathbf{r} = r\cos\theta\,\hat{\mathbf{x}} + r\sin\theta\,\hat{\mathbf{y}} \quad (44)$$
$$r = 1 \quad (45)$$

With $\mathbf{F} = (-y, x)$, this means

$$\mathbf{F} = -r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{y}} \quad (46)$$
$$d\mathbf{r} = -r\sin\theta\,d\theta\,\hat{\mathbf{x}} + r\cos\theta\,d\theta\,\hat{\mathbf{y}} \quad (47)$$

Finally,

$$W = \int \mathbf{F}\cdot d\mathbf{r} = \int (-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})\cdot(-\sin\theta\,d\theta\,\hat{\mathbf{x}} + \cos\theta\,d\theta\,\hat{\mathbf{y}}) \quad (48)$$
$$W = \int_0^{\pi/2} \sin^2\theta + \cos^2\theta\,d\theta = \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2} \quad (49)$$

7. A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius r_o with angular velocity ω_o, but now I pull the string down through the hole until a length r remains between the hole and the particle. **(a)** What is the particle's angular velocity now? **(b)** Assuming that I pull the string so slowly that we can approximate the particle's path by a circle of slowly shrinking radius, calculate the work I did in pulling the string. **(c)** Compare your answer to part (b) with the particle's gain in kinetic energy.

Solution: **(a)** From conservation of angular momentum,

$$L_i = L_f \qquad \Longrightarrow \qquad mr_o^2\omega_o = mr^2\omega \qquad \Longrightarrow \qquad \omega = \omega_o\left(\frac{r_o}{r}\right)^2 \quad (50)$$

(b) We should calculate the work done in moving from r_o to r. We'll need the net force, which must equal the centripetal force $\mathbf{F} = -(mv^2/r)\,\hat{\mathbf{r}} = -mr\omega^2\,\hat{\mathbf{r}}$.

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int -mr\omega^2\, \hat{\mathbf{r}} \cdot d\mathbf{r} = -\int_{r_o}^{r} mr\omega^2\, dr \tag{51}$$

Noting $\omega = \omega_o \left(\frac{r_o}{r}\right)^2$,

$$W = -\int_{r_o}^{r} mr\omega_o^2 \left(\frac{r_o^4}{r^4}\right) dr = -m\omega_o^2 r_o^4 \int_{r_o}^{4} \frac{1}{r^3}\, dr = m\omega_o^2 r_o^4 \left(\frac{1}{2r^2}\right)\Big|_{r_o}^{r} = \frac{1}{2} m\omega_o^2 r_o^4 \left(\frac{1}{r^2} - \frac{1}{r_o^2}\right)$$

$$W = \frac{1}{2} m\omega_o^2 r_o^2 \left[\left(\frac{r_o}{r}\right)^2 - 1\right] \tag{52}$$

(c) We should calculate the gain in kinetic energy.

$$W = \Delta T = \frac{1}{2} mr^2\omega - \frac{1}{2} mr_o^2\omega_o^2 = \frac{1}{2} m \left(r^2\omega_o^2 \left(\frac{r_o^4}{r^4}\right) - r_o^2\omega_o^2\right) = \frac{1}{2} mr_o^2\omega_o^2 \left[\left(\frac{r_o}{r}\right)^2 - 1\right] \tag{53}$$

$\Delta T = W$, as expected.

8. Consider a small frictionless puck perched at the top of a fixed sphere of radius R. If the puck is given a tiny nudge so that it begins to slide down, through what vertical height will it descend before it leaves the sphere? [*Hint:* Use conservation of energy to find the puck's speed as a function of its height, then use Newton's second law to find the normal force of the sphere on the puck. At what value of this normal force does the puck leave the sphere?]

Solution: Let θ be the angular position of the puck on the sphere, with $\theta = 0$ when the puck is a the top of the sphere. At angle θ, the puck will have height $R\cos\theta$ above the ground. With $U = 0$ at ground level, conservation of energy gives

$$mgR = \frac{1}{2} mv^2 + mgR\cos\theta \tag{54}$$

$$v^2 = 2gR\,(1 - \cos\theta) \tag{55}$$

The normal force at an angle θ is the component of the puck's weight acting in the radial direction, and must equal the centripetal force.

$$N = mg\cos\theta = \frac{mv^2}{R} \qquad \Longrightarrow \qquad v^2 = gR\cos\theta \tag{56}$$

Comparing equations 55 and 56, we must have

$$2(1-\cos\theta) = \cos\theta \tag{57}$$

$$3\cos\theta = 2 \tag{58}$$

$$\theta = \arccos\frac{2}{3} \approx 48.2^\circ \tag{59}$$

9. A mass m is in a uniform gravitational field, which exerts the usual force $F = mg$ vertically down, but with g varying according to time, $g = g(t)$. Choosing axes with y measured vertically up and defining $U = mgy$ as usual, show that $\mathbf{F} = -\boldsymbol{\nabla} U$ as usual, but, by differentiating $E = \frac{1}{2}mv^2 + U$ with respect to t, show that E is not conserved.

Solution: With $U = mgy$, we can find the force readily.

$$\mathbf{F} = -\boldsymbol{\nabla} U = -\frac{\partial}{\partial y}(mgy)\,\hat{\mathbf{y}} = -mg\,\hat{\mathbf{y}} \tag{60}$$

As for the energy, noting $g = g(t)$

$$\frac{\partial E}{\partial t} = \frac{1}{2}m(2v\dot{v}) + mg\dot{y} + m\dot{g}y = mva + mgv + my\dot{g} = v(ma + mg) + my\dot{g} \tag{61}$$

But we know $ma = F = -mg$, so

$$\frac{\partial E}{\partial t} = my\dot{g} \neq 0 \tag{62}$$

Energy is conserved only if the gravitational field is time-independent.

10. Consider a mass m on the end of a spring of force constant k and constrained to move along the horizontal x axis. If we place the origin at the spring's equilibrium position, the potential energy is $\frac{1}{2}kx^2$. At time $t = 0$ the mass is sitting at the origin and is given a sudden kick to the right so that it moves out to a maximum displacement at $x_{\text{max}} = A$ and then continues to oscillate about the origin. **(a)** Write down the equation for conservation of energy and solve it to give the mass's velocity $\dot{x}$ in terms of the position x and the total energy E. **(b)** Show that $E = \frac{1}{2}kA^2$, and use this to eliminate E from your expression for $\dot{x}$. Use this result in $t = \int dx'/\dot{x}(x')$ (4.58 in your textbook), to find the time for the mass to move from the origin out to a position x. **(c)** Solve the result of part (b) to give x as a function of t and show that the mass executes simple harmonic motion with period $2\pi\sqrt{m/k}$.

Solution: **(a)** Start with the energy of the oscillator

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{63}$$

$$\dot{x} = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2} \tag{64}$$

(b) At the maximum extent of the mass, we know $x = A$ and $\dot{x} = 0$, so

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \tag{65}$$

$$\implies \qquad \dot{x} = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2} = \sqrt{\frac{kA^2}{m} - \frac{k}{m}x^2} = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \tag{66}$$

Using equation 4.58,

$$t = \int \frac{dx}{\dot{x}} = \int_0^x \sqrt{\frac{m}{k}} \frac{1}{\sqrt{A^2 - x^2}}\, dx = \sqrt{\frac{m}{k}} \int_0^x \frac{1}{\sqrt{A^2 - x^2}}\, dx \tag{67}$$

$$t = \sqrt{\frac{m}{k}} \arcsin\left(\frac{x}{A}\right)\Bigg|_0^x = \sqrt{\frac{m}{k}} \arcsin\left(\frac{x}{A}\right) \tag{68}$$

(c) Invert the arcsin and solve for $x(t)$:

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{69}$$

This is indeed simple harmonic motion with $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$.