University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 4

due 21 Sept 2018

Instructions:

1. Answer all questions below (bonus questions optional).
2. Show your work for full credit.
3. All problems are due by 11:59pm on 21 Sept 2018.
4. You may collaborate, but everyone must turn in their own work.

1. The potential energy of two atoms in a molecule can sometimes be approximated by the Morse potential,

$$U(r) = A\left[\left(e^{-(r-R)/S} - 1\right)^2 - 1\right] \tag{1}$$

where r is the distance between the two atoms and A, R, and S are positive constants with $S \ll R$. **(a)** Sketch this function for $0 < r < \infty$. **(b)** Find the equilibrium separation r_o, at which $U(r)$ is minimum. **(c)** Now write $r = r_o + x$ so that x is the displacement from equilibrium, and show that for small displacements, U has the approximate form $U = \text{const} + \frac{1}{2}kx^2$. That is, Hooke's law applies. **(d)** What is the force constant k? **(bonus +5)** Interpret R, S, and A physically in terms of characteristic properties of diatomic molecules. *Hint: the Morse potential is well known.*

2. An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R, wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium, the mass hangs a distance l_o vertically below the edge of the cylinder. **(a)** Find the potential energy if the pendulum has swung to an angle φ from the vertical. **(b)** Show that for small angles, it can be written in Hooke's law form $U = \frac{1}{2}k\varphi^2$. **(c)** Comment on the value of k. *Hint: Draw a figure, understand the geometry. At an angle φ, what length of rope is wrapped around the cylinder compared to when $\varphi = 0$? The remaining length helps you find the height.*

3. A practical sort of problem. You are told that at known positions x_1 and x_2, an oscillating mass m has speeds v_1 and v_2. What are the amplitude and angular frequency of the oscillations?

4. The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_o\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right) \tag{2}$$

for $0 < r < \infty$, with U_o, R, and λ all positive constants. **(a)** Find the equilibrium position r_o. **(b)** Let x be the distance from equilibrium and show that, for small x, the PE has the form

$U = \text{const} + \frac{1}{2}kx^2$. **(c)** What is the angular frequency of small oscillations? *Hint: for small displacements, Taylor expand U about r_o and focus on the second-order term. The constant term can be defined to zero by a suitable choice of zero potential energy. What must be true of the first-order term in equilibrium?*

5. Consider a two-dimensional anisotropic oscillator with motion given by

$$x(t) = A_x \cos(\omega_x t) \tag{3}$$

$$y(t) = A_y \cos(\omega_y t - \delta) \tag{4}$$

(a) Prove that if the ratio of frequencies is rational (that is, $\omega_x/\omega_y = p/q$ where p and q are integers) then the motion is periodic. What is the period? **(b)** Prove that if the same ratio is irrational, the motion never repeats itself.

6. A damped oscillator satisfies the equation

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{5}$$

where $F_{\text{damp}} = -b\dot{x}$ is the damping force. Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (by straightforward differentiation), and, with the help of the equation above, show that dE/dT is (minus) the rate at which energy is dissipated by F_{damp}.

7. An undamped oscillator has period $\tau_o = 1.000\,\text{s}$, but I now add a little damping so that its period changes to $\tau_1 = 1.001\,\text{s}$. **(a)** What is the damping factor β? **(b)** By what factor will the amplitude of oscillation decrease after 10 cycles? **(c)** Which effect of damping would be more noticeable, the change in period or the decrease in amplitude? Justify your answer.

8. The solution for $x(t)$ for a driven, underdamped oscillator is most conveniently found in the form

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)\right] \tag{6}$$

Solve the equation above and the corresponding expression for $\dot{x}$, to give the coefficients B_1 and B_2 in terms of A, δ, and the initial position and velocity x_o and v_o. You should reproduce the expressions given in Example 5.3 in your textbook.

9. We know that if the driving frequency ω is varied, the maximum response (A^2) of a driven damped oscillator occurs at $\omega \approx \omega_o$ (if the natural frequency is ω_o, and the damping constant $\beta \ll \omega_o$). Show that A^2 is equal to half its maximum value when $\omega \approx \omega_o \pm \beta$, so that the full width at half maximum is just 2β. [*Hint: be careful with your approximations. For instance, it is fine to say $\omega + \omega_o \approx 2\omega_o$, but you certainly can't say $\omega - \omega_o \approx 0$.*]

10. Another interpretation of the Q of a resonance comes from the following: Consider the motion of a driven damped oscillator after any transients have died out, and suppose that it is being driven close to resonance so you can set $\omega = \omega_o$. **(a)** Show that the oscillator's total energy (kinetic plus potential) is $E = \frac{1}{2}m\omega^2 A^2$. **(b)** Show that the energy ΔE_{dis} dissipated during one cycle by the damping force F_{damp} is $2\pi m\beta\omega A^2$. (Remember that the rate at which a force does work is Fv.) **(c)** Hence show that Q is 2π times the ratio of $E/\Delta E_{\text{dis}}$.

11. BONUS +10: Consider a cart on a spring with natural frequency $\omega_o = 2\pi$, which is released from rest at $x_o = 1$ and $t = 0$. Using appropriate software, plot the position $x(t)$ for $0 < t < 2$ and for damping constants $\beta = 0, 1, 2, 4, 6, 2\pi, 10, 20$. [Remember that $x(t)$ is given by different formulas for $\beta < \omega_o$, $\beta = \omega_o$, and $\beta > \omega_o$.]