University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

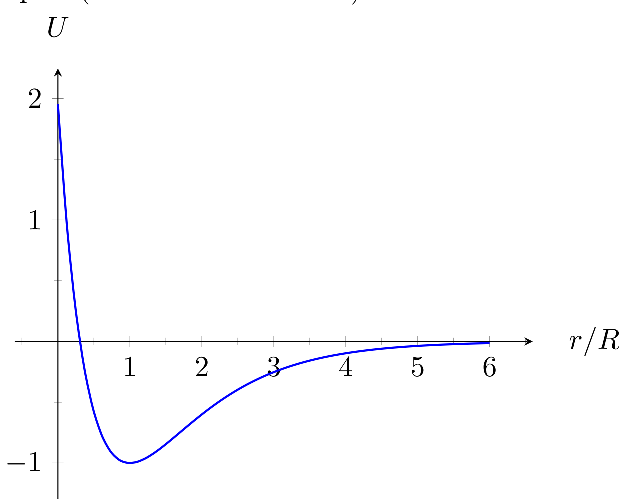
Problem Set 4 Solutions

1. The potential energy of two atoms in a molecule can sometimes be approximated by the Morse potential,

$$U(r) = A\left[\left(e^{-(r-R)/S} - 1\right)^2 - 1\right] \tag{1}$$

where r is the distance between the two atoms and A, R, and S are positive constants with $S \ll R$. **(a)** Sketch this function for $0 < r < \infty$. **(b)** Find the equilibrium separation r_o, at which $U(r)$ is minimum. **(c)** Now write $r = r_o + x$ so that x is the displacement from equilibrium, and show that for small displacements, U has the approximate form $U = \text{const} + \frac{1}{2}kx^2$. That is, Hooke's law applies. **(d)** What is the force constant k? **(bonus +5)** Interpret R, S, and A physically in terms of characteristic properties of diatomic molecules. *Hint: the Morse potential is well known.*

Solution: When $r = 0$, we have $U = A\left[(e^{R/S} - 1)^2 - 1\right]$, which is large and positive since $R \gg S$. When r is large, U is negative since the term in () is less than one, and as $r \to \infty$, $U \to 0$. Here is a plot (with $A = R = S = 1$):



The minimum value of U occurs at $r = R$ with $U(R) = -A$, meaning the equilibrium separation is $r_o = R$.

$$\frac{dU}{dr} = 2A\left(e^{-(r-R)/S} - 1\right)\left(\frac{R-r}{S}e^{-(r-R)/S}\right) \tag{2}$$

$$= 2A\left(\frac{R-r}{S}\right)\left(e^{-2(r-R)/S} - e^{-(r-R)/S}\right) = 0 \tag{3}$$

$$\implies \qquad r = R \tag{4}$$

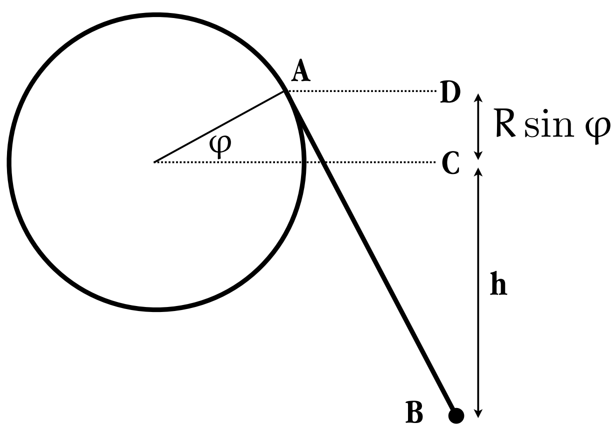
For small displacements around r_o, set $r = R + x$ where $x << R$ and Taylor expand about R:

$$U(R+x) = A\left[\left(\left\{1 - \frac{x}{S} + \ldots\right\} - 1\right)^2 - 1\right] \approx -A + A\left(\frac{x}{S}\right)^2 = \text{const } + \frac{1}{2}kx^2 \tag{5}$$

Where $k = 2A/S^2$. The term in curly brackets {} is the first two terms in the Taylor expansion of $e^{-(r-R)/S} = e^{-x/S}$.

2. An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R, wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium, the mass hangs a distance l_o vertically below the edge of the cylinder. **(a)** Find the potential energy if the pendulum has swung to an angle φ from the vertical. **(b)** Show that for small angles, it can be written in Hooke's law form $U = \frac{1}{2}k\varphi^2$. **(c)** Comment on the value of k. *Hint: Draw a figure, understand the geometry. At an angle φ, what length of rope is wrapped around the cylinder compared to when $\varphi = 0$? The remaining length helps you find the height.*

Solution: See the figure below. The PE is $U = -mgh$, where h is the height of the mass with respect to the center of the cylinder. As the pendulum swings through an angle φ, a length of rope $R\varphi$ unwinds off of the cylinder. (When it swings back the other way past equilibrium, a length $R\varphi$ winds around the cylinder).



Thus, the free length of the string not touching the cylinder at an angle φ is the length AB or $l_o + R\varphi$. The height BD is that length times $\cos\varphi$, i.e., $(l_o + R\varphi)\cos\varphi$. The last distance we need, CD, is $R\sin\varphi$, and we can find h since it must equal $BC - CD$, or

$$h = l_o \cos\varphi + R(\varphi\cos\varphi - \sin\varphi) \tag{6}$$

Thus,

$$U = -mgh = -mg\left[l_o \cos\varphi + R\left(\varphi\cos\varphi - \sin\varphi\right)\right] \tag{7}$$

If we assume φ is small, then $\cos\varphi \approx 1 - \varphi^2/2$ and $\sin\varphi \approx \varphi$:

$$U \approx -mg\left[l_o - \frac{1}{2}l_o\varphi^2 + R\left(\varphi(1-\frac{1}{2}\varphi^2) - \varphi\right)\right] \quad = -mg\left[l_o - \frac{1}{2}l_o\varphi^2 + R(\varphi - \frac{1}{2}\varphi^3 - \varphi)\right] \tag{8}$$

Neglecting the term of order φ^3,

$$U \approx -mgl_o - \frac{1}{2}mgl_o\varphi^2 = \text{const } + \frac{1}{2}k\varphi^2 \tag{9}$$

The constant $k = mgl_o$ is just what you would get for a pendulum of length l_o. The effect of the cylinder is apparent only at higher orders than φ^2 - for small oscillations, wrapping the string around the cylinder makes no difference. That is not surprising - for small angles, the fractional change in the string's length is negligible. Note that if we did keep the third order term in φ, the potential energy would be an odd function rather than an even function, meaning it is slightly easier to unwrap string than to wrap string.

3. A practical sort of problem. You are told that at known positions x_1 and x_2, an oscillating mass m has speeds v_1 and v_2. What are the amplitude and angular frequency of the oscillations?

Solution: This means we have two expressions for the energy of the oscillator, which we know is constant.

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 \tag{10}$$
$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2 \tag{11}$$

Subtracting the two equations, we find

$$m\left(v_1^2 - v_2^2\right) = k\left(x_2^2 - x_1^2\right) \tag{12}$$

Rearranging,

$$\frac{k}{m} = \omega^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2} \tag{13}$$

Going back to the first equation, and using the result above,

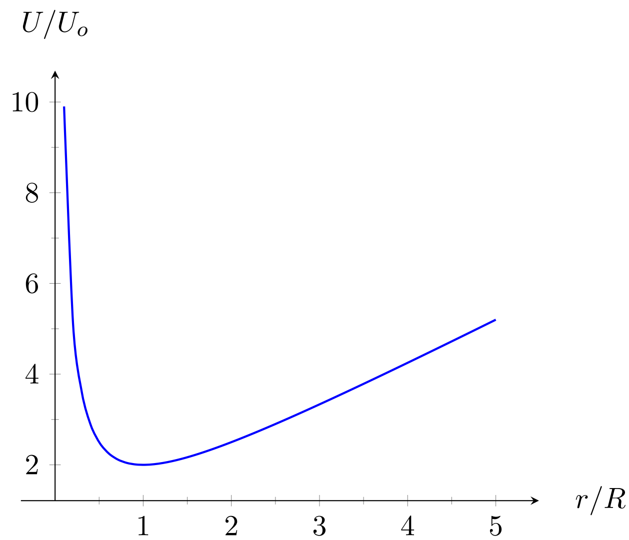
$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \left(\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}\right) v_1^2 + x_1^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2} \tag{14}$$

4. The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_o\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right) \tag{15}$$

for $0 < r < \infty$, with U_o, R, and λ all positive constants. **(a)** Find the equilibrium position r_o. **(b)** Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const} + \frac{1}{2}kx^2$. **(c)** What is the angular frequency of small oscillations? *Hint: for small displacements, Taylor expand U about r_o and focus on the second-order term. The constant term can be defined to zero by a suitable choice of zero potential energy. What must be true of the first-order term in equilibrium?*

Solution: A plot is always handy so we have an idea of what we are dealing with. We'll plot r/R on the x axis and U/U_o on the y axis (so units don't matter) with $\lambda = 1$.



As one would expect from the form of $U(r)$, the $1/r$ term dominates for small r, while at large r the behavior is linear. This implies that there is a minimum in between, which we observe in the plot. The equilibrium position is when $\partial U/\partial r = 0$ and $\partial^2 U/\partial r^2 > 0$.

$$\frac{\partial U}{\partial r} = \frac{U_o}{R} + \lambda^2 U_o\left(\frac{-R}{r}\right) = U_o\left(\frac{1}{R} - \lambda^2\frac{R}{r^2}\right) = 0 \tag{16}$$

$$\implies \qquad \frac{1}{R} = \lambda^2\frac{R}{r^2} \implies \qquad \implies r = \pm\lambda R \tag{17}$$

Checking the second derivative,

$$\frac{\partial^2 U}{\partial r^2} = \frac{2U_o \lambda^2 R}{r^3} \tag{18}$$

Clearly $\partial^2 U/\partial r^2 > 0$ for all $r > 0$, so the stable equilibrium point must be at $r_o = +\lambda R$. The solution at $r = -\lambda R$ is an unstable equilibrium by the second derivative test. More to the point, $r < 0$ is unphysical, since r is a distance from the origin and therefore always positive. For small displacements about a stable equilibrium, we have already established that $k = U''(r_o)$. Using the result above,

$$k = U''(r_o) = \frac{2U_o \lambda^2 R}{r_o^3} = \frac{2U_o \lambda^2 R}{\lambda^3 R^3} = \frac{2U_o}{\lambda R^2} \tag{19}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2U_o}{\lambda m R^2}} \tag{20}$$

5. Consider a two-dimensional anisotropic oscillator with motion given by

$$x(t) = A_x \cos(\omega_x t) \tag{21}$$

$$y(t) = A_y \cos(\omega_y t - \delta) \tag{22}$$

(a) Prove that if the ratio of frequencies is rational (that is, $\omega_x/\omega_y = p/q$ where p and q are integers) then the motion is periodic. What is the period? **(b)** Prove that if the same ratio is irrational, the motion never repeats itself.

Solution: **(a)** From the given cos forms for x and y, it is clear both are periodic with $T_x = 2\pi/\omega_x$ and $T_y = 2\pi/\omega_y$. Periodicity requires that the periods T_x and T_y match eventually, meaning that for two numbers p and q we require

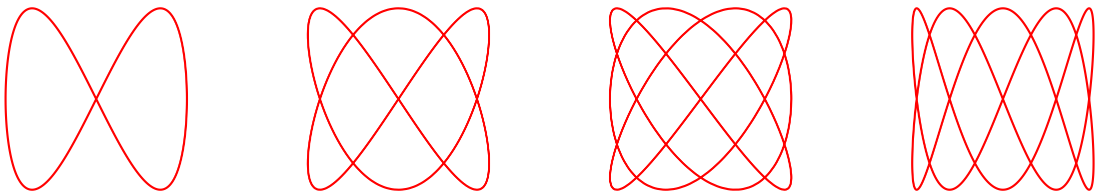
$$pT_x = qT_y \tag{23}$$

The period of the overall motion is for the smallest (p, q) that satisfy the equation above. Clearly, this only works if $T_x = (p/q)T_y$, which is only possible if p and q are integers, $\{p, q\} \in \mathcal{Z}$.

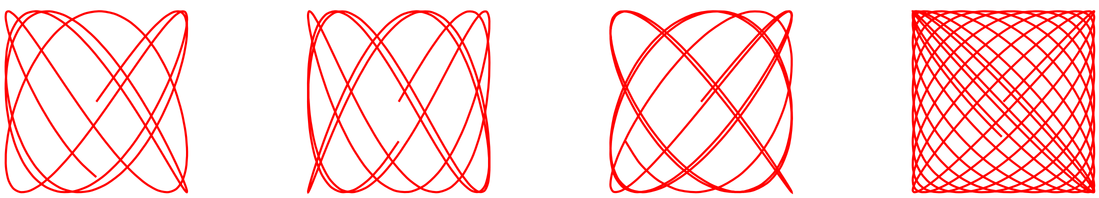
(b) If p/q is irrational, p and q have no least common multiple that is an integer, and our condition for repetition above cannot be met. If p/q is irrational, then if one goes through p revolutions of 2π for the x motion, there are $2\pi q$ revolutions of the y motion that is not a whole number of revolutions and the motion cannot repeat.

Though not a rigorous proof, a visual demonstration is illustrative. Here are plots for rational $p : q$

in ratios $(1:2),(2:3),(3:4),(2:5)$ from left to right. Notice how there are p repeats along x and q repeats along y.



Now look at the some plots for irrational values of p/q, with $p:1=(1:\sqrt{2}),(1:\sqrt{3}),(\sqrt{2}:\sqrt{3}),(3:\pi)$ from left to right. Even though we quadrupled the range of times plotted, the motion still does not repeat – the curve never closes back on itself.



These types of curves are known as *Lissajous curves* (https://en.wikipedia.org/wiki/Lissajous_curve), and can be generated on an oscilloscope fairly easily with a pair of function generators.

6. A damped oscillator satisfies the equation

$$m\ddot{x}+b\dot{x}+kx=0 \tag{24}$$

where $F_{\text{damp}}=-b\dot{x}$ is the damping force. Find the rate of change of the energy $E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$ (by straightforward differentiation), and, with the help of the equation above, show that dE/dT is (minus) the rate at which energy is dissipated by F_{damp}.

Solution: We know the energy is

$$E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2 \tag{25}$$

Differentiating with respect to time,

$$\frac{dE}{dt}=\frac{1}{2}m\left(2\dot{x}\ddot{x}\right)+kx\dot{x}=\dot{x}\left(m\ddot{x}+kx\right) \tag{26}$$

But, we know $m\ddot{x}=-kx-F_{\text{damp}}=-kx-b\dot{x}$, so $m\ddot{x}+kx=-b\dot{x}$.

$$\frac{dE}{dt}=\dot{x}\left(m\ddot{x}+kx\right)=\dot{x}\left(-b\dot{x}\right)=-vF_{\text{damp}} \tag{27}$$

Since vF_{damp} is the rate at which the damping force does work, so we have the desired result.

7. An undamped oscillator has period $\tau_o = 1.000\,\mathrm{s}$, but I now add a little damping so that its period changes to $\tau_1 = 1.001\,\mathrm{s}$. **(a)** What is the damping factor β? **(b)** By what factor will the amplitude of oscillation decrease after 10 cycles? **(c)** Which effect of damping would be more noticeable, the change in period or the decrease in amplitude? Justify your answer.

Solution: Note $\tau_o = 2\pi/\omega_o$ and $\tau_1 = 2\pi/\omega$. For a lightly damped oscillator, we know $\omega_1^2 = \omega_o^2 - \beta^2$, so $\beta^2 = \omega_o^2 - \omega^2 = \omega^2\left(1 - \omega_1^2/\omega_o^2\right)$. Thus,

$$\beta = \omega_o\sqrt{1 - \frac{\omega_1^2}{\omega_o^2}} = \omega_o\sqrt{1 - \frac{\tau_o^2}{\tau_1^2}} = \omega_o\sqrt{1 - \frac{1}{1.001^2}} \approx 0.0447\omega_o = 0.281\,\mathrm{s}^{-1} \tag{28}$$

Since $\beta \approx 0.0447\omega_o$, our claim of light damping is justified. At a time $t = 10\tau_1 \approx 10\tau_o$, the amplitude changes by a factor

$$\frac{A}{A_o} = e^{-\beta t} = e^{-10\beta\tau_o} \approx 0.060 \tag{29}$$

The amplitude changes by about 1 part in 17 ($1/0.060 \sim 17$), whereas the period changes by only 1 part in 1000 (0.1%), clearly the change in amplitude is more noticeable.

8. The solution for $x(t)$ for a driven, underdamped oscillator is most conveniently found in the form

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] \tag{30}$$

Solve the equation above and the corresponding expression for $\dot{x}$, to give the coefficients B_1 and B_2 in terms of A, δ, and the initial position and velocity x_o and v_o. You should reproduce the expressions given in Example 5.3 in your textbook.

Solution: Just have to do the math.

$$\begin{aligned}
x(t) &= A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] && (31)\\
x(0) &= A\cos\delta + B_1 \equiv x_o && (32)\\
\dot{x}(t) &= -\omega A\sin(\omega t - \delta) - \beta e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] && (33)\\
&\qquad + e^{-\beta t}\left[-\omega_1 B_1\sin(\omega_1 t) + \omega_1 B_2\cos(\omega_1 t)\right] && (34)\\
\dot{x}(0) &= -\omega A\sin -\delta - \beta B_1 + \omega_1 B_2 = \omega A\sin\delta - \beta B_1 + \omega_1 B_2 \equiv v_o && (35)\\
\Longrightarrow B_1 &= x_o - A\cos\delta && (36)\\
\Longrightarrow B_2 &= \frac{1}{\omega_1}\left(v_o - \omega A\sin\delta + \beta B_1\right) && (37)
\end{aligned}$$

This matches equation 5.70 in example 5.3 in the text.

9. We know that if the driving frequency ω is varied, the maximum response (A^2) of a driven damped oscillator occurs at $\omega \approx \omega_o$ (if the natural frequency is ω_o, and the damping constant $\beta \ll \omega_o$). Show that A^2 is equal to half its maximum value when $\omega \approx \omega_o \pm \beta$, so that the full width at half maximum is just 2β. [*Hint: be careful with your approximations. For instance, it is fine to say* $\omega + \omega_o \approx 2\omega_o$*, but you certainly can't say* $\omega - \omega_o \approx 0$.]

Solution: The response is given by

$$A^2 = \frac{f_o^2}{(\omega_o - \omega)^2 + 4\beta^2\omega^2} \tag{38}$$

For $\beta \ll \omega$, the maximum response comes when $\omega \approx \omega_o$, where

$$A^2_{\text{max}} \approx \frac{f_o^2}{4\beta^2\omega^2} \approx \frac{f_o^2}{4\beta^2\omega_o^2} \tag{39}$$

When A^2 is half this value, the denominator in the original expression should be twice as big as in the expression above, $8\beta^2\omega_o^2$. At some frequency $\omega \not\approx \omega_o$, this means

$$8\beta^2\omega_o^2 = (\omega_o - \omega)^2 + 4\beta^2\omega^2 \tag{40}$$

$$4\beta^2\omega^2 = (\omega_o - \omega)^2 = (\omega_o - \omega)^2 (\omega_o + \omega)^2 \tag{41}$$

While we can't say $\omega_o - \omega \approx 0$, we can say $\omega_o + \omega \approx 2\omega_o$, so

$$4\beta^2\omega^2 = (\omega_o - \omega)^2 (\omega_o + \omega)^2 \approx (2\omega_o)^2 (\omega_o - \omega)^2 = 4\omega_o^2 (\omega_o - \omega)^2 \tag{42}$$

$$\implies \qquad \omega_o - \omega \approx \pm\beta \tag{43}$$

Thus, the half maxima occur at $\omega = \omega_o \pm \beta$, so the full-width at half-maximum (FWHM) is 2β.

10. Another interpretation of the Q of a resonance comes from the following: Consider the motion of a driven damped oscillator after any transients have died out, and suppose that it is being driven close to resonance so you can set $\omega = \omega_o$. **(a)** Show that the oscillator's total energy (kinetic plus potential) is $E = \frac{1}{2}m\omega^2A^2$. **(b)** Show that the energy ΔE_{dis} dissipated during one cycle by the damping force F_{damp} is $2\pi m\beta\omega A^2$. (Remember that the rate at which a force does work is Fv.) **(c)** Hence show that Qs is 2π times the ratio of $E/\Delta E_{\text{dis}}$.

Solution: Since $x(t) = A\cos(\omega t - \delta)$, the total energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t - \delta) + \frac{1}{2}kA^2\sin^2(\omega t - \delta) \tag{44}$$

Because $\omega \approx \omega_o$, we can say $k = m\omega_o^2 \approx m\omega^2$, and

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t - \delta) + \frac{1}{2}kA^2 \sin^2(\omega t - \delta) \tag{45}$$

$$= \frac{1}{2}m\omega^2 A^2 \left[\cos^2(\omega t - \delta) + \sin^2(\omega t - \delta)\right] = \frac{1}{2}m\omega^2 A^2 \tag{46}$$

The rate at which the damping force dissipates energy is $F_{\text{damp}}v = bv^2 = 2m\beta v^2 = \frac{dE}{dt}$ noting $\beta = b/m$. Integrating over one period, with $v^2 = \omega^2 A^2 \sin^2(\omega t - \delta)$

$$\Delta E_{\text{dis}} = \int_0^\tau 2m\beta v^2\, dt = 2m\beta\omega^2 A^2 \int_0^\tau \sin^2(\omega t - \delta)\, dt = 2m\beta\omega^2 A^2 \int_0^\tau \frac{1}{2}\left[1 - \sin(2\omega t - 2\delta)\right]\, dt \tag{47}$$

For the last step, we noted $2\sin^2\theta = 1 - \sin 2\theta$. Since the integral of $\sin 2\theta$ over one complete cycle is zero, we have

$$\Delta E_{\text{dis}} = 2m\beta\omega^2 A^2 \int_0^\tau \frac{1}{2}\, dt = 2m\beta\omega^2 A^2 \left(\frac{\tau}{2}\right) = 2m\beta\omega^2 A^2 \left(\frac{\pi}{\omega}\right) = 2\pi m\beta\omega A^2 \tag{48}$$

We could have also noted that the average value of $\sin^2\theta$ over one cycle is $1/2$ and saved a step. Combining the two results, and noting again $\omega \approx \omega_o$,

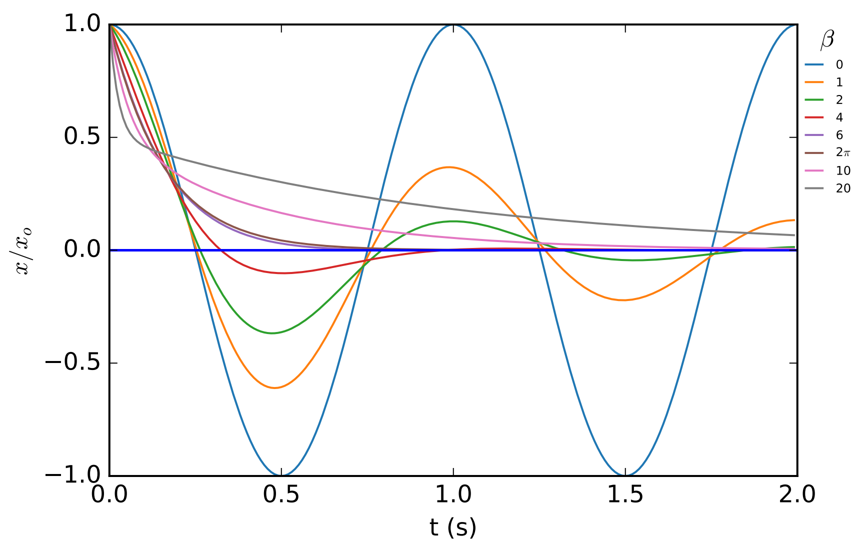
$$\frac{E}{\Delta E_{\text{dis}}} = \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m\beta\omega A^2} = \frac{\omega}{4\pi\beta} \approx \frac{\omega_o}{4\pi\beta} = \frac{Q}{2\pi} \tag{49}$$

Another way of stating this is

$$Q = 2\pi\frac{\text{energy stored}}{\text{energy dissipated per cycle}} = \omega_o\frac{\text{energy stored}}{\text{power loss}} = \frac{\omega_o}{2\beta} = \frac{\text{resonance frequency}}{\text{FWHM of resonance curve}} \tag{50}$$

11. BONUS +10: Consider a cart on a spring with natural frequency $\omega_o = 2\pi$, which is released from rest at $x_o = 1$ and $t = 0$. Using appropriate software, plot the position $x(t)$ for $0 < t < 2$ and for damping constants $\beta = 0, 1, 2, 4, 6, 2\pi, 10, 20$. [Remember that $x(t)$ is given by different formulas for $\beta < \omega_o$, $\beta = \omega_o$, and $\beta > \omega_o$.]

Solution: On the next page is some python code that does the job, and below is a plot that generates.



```python
import math
import matplotlib.pyplot as plt
import numpy as np

def x(b,wo):
    dt = 0.01 #time step for simulation
    ttemp=0
    t=[]
    x=[]
    xtemp = 0

    if (b<wo): #weak damping
        while ttemp<2.0:
                xtemp = math.exp(-b*ttemp)*math.cos(math.sqrt(wo*wo-b*b)*ttemp)
                x.append(xtemp)
                t.append(ttemp)
                ttemp+=dt
    if (b==wo): #critical damping
        while ttemp<2.0:
                xtemp = math.exp(-b*ttemp) + ttemp*math.exp(-b*ttemp)
                x.append(xtemp)
                t.append(ttemp)
                ttemp+=dt
    if (b>wo): #strong damping
        while ttemp<2.0:
                xtemp = 0.5*math.exp(-(b-math.sqrt(b*b-wo*wo))*ttemp)
+ 0.5*math.exp(-(b+math.sqrt(b*b-wo*wo))*ttemp)
                x.append(xtemp)
                t.append(ttemp)
                ttemp+=dt

    return (x,t)

b= [0,1,2,4,6,6.2832,10,20] #various values of beta and corresponding labels
labels = ['0','1','2','4','6',r'2$\pi$','10','20']

wo=2*3.14159 #osc freq without damping

#presuming initial position is xo=0 at t=0

colors = ['#1f77b4', '#ff7f0e', '#2ca02c', '#d62728',
                '#9467bd', '#8c564b', '#e377c2', '#7f7f7f']
#just to cycle through colors without repeats.

for j in range(len(b)): #calculate trajectories for different betas
        trajectory,t = x(b[j],wo)
        plt.plot(t,trajectory,label=labels[j],color=colors[j])
        plt.axhline(linewidth=1)

plt.xlabel('t␣(s)')
plt.ylabel(r'$x/x_o$')

plt.legend(title=r'$\beta$',fancybox=True,frameon=0,bbox_to_anchor=(1.0,1.0),
                loc="best", borderaxespad=0,prop={'size':6})
plt.tight_layout(pad=7)

plt.savefig('5-31.pdf',format='pdf')
plt.show()  #write to screen
```