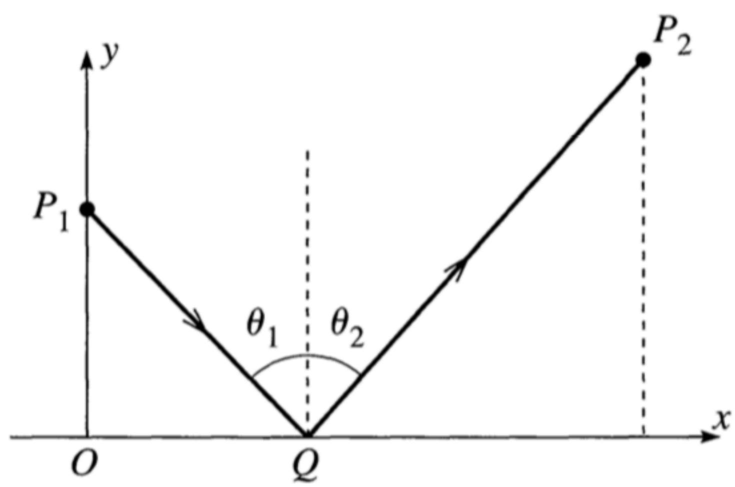
University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 5 Solutions

1. Consider a ray of light traveling in a vacuum from point P_1 to P_2 by way of point Q on a plane mirror, as shown below. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, $\theta_1 = \theta_2$. [*Hints:* let the mirror lie in the xz plane, and let P_1 lie on the y axis at $(0, y_1, 0)$ and P_2 in the xy plane at $(x_2, y_2, 0)$. Finally, let $Q = (x, 0, z)$. Calculate the time for light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]



Solution: With the points given, we want to find the distances P_1Q and QP_2.

$$P_1Q = \sqrt{x^2 + y_1^2 + z^2} \tag{1}$$

$$QP_2 = \sqrt{(x_2 - x)^2 + y_2^2 + z^2} \tag{2}$$

Given the velocity is $v = c$, the total time to traverse the two segments is then

$$t = \frac{P_1Q}{c} + \frac{QP_2}{c} = \frac{1}{c}\sqrt{x^2 + y_1^2 + z^2} + \frac{1}{c}\sqrt{(x_2 - x)^2 + y_2^2 + z^2} \tag{3}$$

Minimizing with respect to z,

$$\frac{\partial t}{\partial z} = \frac{z}{c\sqrt{x^2+y_1^2+z^2}} + \frac{z}{(x_2-x)^2+y_2^2+z^2} = z\left(\frac{1}{c\sqrt{x^2+y_1^2+z^2}} + \frac{1}{(x_2-x)^2+y_2^2+z^2}\right) = 0 \tag{4}$$

Clearly, the minimum occurs only when $z = 0$, which means that Q, P_1, and P_2 all lie within the xy plane. Referencing the figure and the definitions of P_1, P_2, and Q, we can see (with $z = 0$) that

$$\sin\theta_1 = \frac{x}{\sqrt{x^2+y_1^2}} = \sin\theta_2 = \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2+z^2}} \tag{5}$$

One can arrive at the same result by noting that we also require $\partial t/\partial x = 0$ to minimize time.

2. Find the equation of the path joining the origin O to the point $P(1,1)$ in the xy plane that makes the integral $\int_O^P (y'^2 + yy' + y^2)\,dx$ stationary.

Solution: The integrand is $f = y'^2 + yy' + y^2$, we can apply the Euler-Lagrange equations.

$$\begin{aligned} \frac{\partial f}{\partial y} &= y' + 2y && (6)\\ \frac{d}{dx}\frac{\partial f}{\partial y'} &= \frac{d}{dx}\big(2y' + y\big) = 2y'' + y' && (7)\\ \implies \qquad 0 &= y' + 2y - \big(2y'' + y'\big) && (8)\\ y'' &= y && (9) \end{aligned}$$

We've encountered this equation before, and we know the solution is $y(x) = Ae^x + Be^{-x}$. We can find A and B from the given points. Since $y(0) = 0$ and $y(1) = 1$, we must have

$$\begin{aligned} 0 &= A + B && (10)\\ 1 &= Ae + B/e = Ae - A/e && (11)\\ A &= \frac{1}{e-1/e} = -B && (12)\\ y(x) &= \frac{e^x}{e-1/e} - \frac{e^{-x}}{e-1/e} = \frac{e^x - e^{-x}}{e-1/e} = \frac{2\sinh x}{2\sinh 1} = \frac{\sinh x}{\sinh 1} && (13) \end{aligned}$$

3. In general the integrand $f(y, y', x)$ whose integral we wish to minimize depends on y, y', and x. There is considerable simplification if f happens to be independent of y, that is, $f = f(y', x)$. Prove that when this happens, the Euler-Lagrange equation reduces to the statement that $\partial f/\partial y' = \text{const}$. Since this is a first-order differential equation for $y(x)$, while the Euler-Lagrange equation is generally second order, this is an important simplification and the result is sometimes called a *first integral* of the Euler-Lagrange equation. In Lagrangian mechanics, we'll see that this simplification arises when a component of momentum is conserved.

Solution: If $\partial f/\partial y = 0$, then the Euler-Lagrange equation reduces to

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{14}$$

which indeed implies $\partial f/\partial y' = \text{const}$.

4. You are given a string of fixed length l with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x axis in such a way as to enclose the maximum area between the string and the x axis. Show that the required shape is a semicircle. The area enclosed is $\int y\,dx$, but show that you can rewrite this in the form $\int_O^l f\,ds$, where s denotes the distance measured along the string from O, where $f = y\sqrt{1-y'^2}$, and y' denotes dy/ds. Since f does not involve the independent variable s explicitly, you can exploit the result of the previous problem.

Solution: The area between the string and the x axis is

$$A = \int y\,dx \tag{15}$$

Since $ds^2 = dx^2 + dy^2$, then with $y' \equiv dy/ds$,

$$dx = \sqrt{ds^2 - dy^2} = ds\sqrt{1-(dy/ds)^2} = ds\sqrt{1-y'^2} \tag{16}$$

The area integral is then

$$A = \int_0^l y\,dx = y\sqrt{1-y'^2}\,ds \tag{17}$$

Our function is thus $f(y, y', x) = y\sqrt{1-y'^2}$. Since this is independent of x, the result of problem 6.20 applies, which we take a detour to prove.

Begin Detour

We know $f = f(y, y')$, then standard calculus tells us

$$df = \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial y'}\,dy' \tag{18}$$

Divide everything by dx, noting $y' = dy/dx$ and $y'' = dy'/dx$:

$$\frac{df}{dx} = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{19}$$

From the Euler-Lagrange equation, we know $\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$, so substituting and using the product rule,

$$\frac{df}{dx} = \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) \tag{20}$$

Since both the leftmost and rightmost sides are derivatives with respect to x, it must be the case that the two derivatives are equal up to an overall constant, so

$$f = y' \frac{\partial f}{\partial y'} + \text{constant} \tag{21}$$

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} \tag{22}$$

End Detour

With this in hand, and "constant" replaced by R,

$$f - y' \frac{\partial f}{\partial y'} = y\sqrt{1 - y'^2} - y' \left(\frac{y(-y')}{\sqrt{1 - y'^2}}\right) = \frac{y(1 - y'^2) + yy'^2}{\sqrt{1 - y'^2}} = \frac{y}{\sqrt{1 - y'^2}} = R \tag{23}$$

$$y^2 = R^2 \left(1 - y'^2\right) \tag{24}$$

$$y'^2 = 1 - \frac{y^2}{R^2} = \left(\frac{dy}{ds}\right)^2 \tag{25}$$

$$ds = \frac{dy}{\sqrt{1 - y^2/R^2}} \tag{26}$$

We can use this to try to figure out x. (Figuring out y then x works just fine as well, but this is shorter.)

$$ds^2 = dx^2 + dy^2 = \frac{dy^2}{1 - y^2/R^2} \tag{27}$$

$$dx^2 = dy^2 \left(\frac{1}{1 - y^2/R^2} - 1\right) = dy^2 \left(\frac{y^2/R^2}{1 - y^2/R^2}\right) \tag{28}$$

$$dx = dy \frac{y}{\sqrt{R^2 - y^2}} \tag{29}$$

$$x = \int \frac{y}{\sqrt{R^2 - y^2}}\, dy = -\sqrt{R^2 - y^2} + \text{constant} \tag{30}$$

Since $x = 0$ when $y = 0$, the constant must be R, so

$$x = -\sqrt{R^2 - y^2} + R \tag{31}$$

$$R^2 = (x - R)^2 + y^2 \tag{32}$$

This is the equation of a semicircle of radius R centered on the point $(R, 0)$, so the string is fixed at $(0, 0)$ and $(2R, 0)$ along the x axis. Thus, a circle encloses the maximum area for a given length of string in flat space.

How does R relate to l? After all, we still require the total length of the curve to be l. It is not clear since l does not appear in our equation. One way we can figure this out is by just asserting that the system has been constrained such that the length of the string is equal to the length of the semicircle, $l = \pi R$. We can also get to the same requirement by finding $y(s)$. Go back to Eq. 26 and integrate both sides,

$$\int ds = \int \frac{dy}{\sqrt{1 - y^2/R^2}} \tag{33}$$

$$s = R \arcsin\left(\frac{y}{R}\right) + C \qquad \text{note } y = 0 \text{ when } s = 0, \text{ so } C = 0 \tag{34}$$

$$y = R \sin\frac{s}{R} \tag{35}$$

Since $y = 0$ when $s = l$ as well, it must be that $l/R = n\pi$ where n is an integer. The length of the semicircle must be half the circumference of a circle of radius R, or $\pi R = l/n$. Since the length of the semicircle must also be the length of the string l, we require $n = 1$. Any other integer n yields a smaller area.