University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

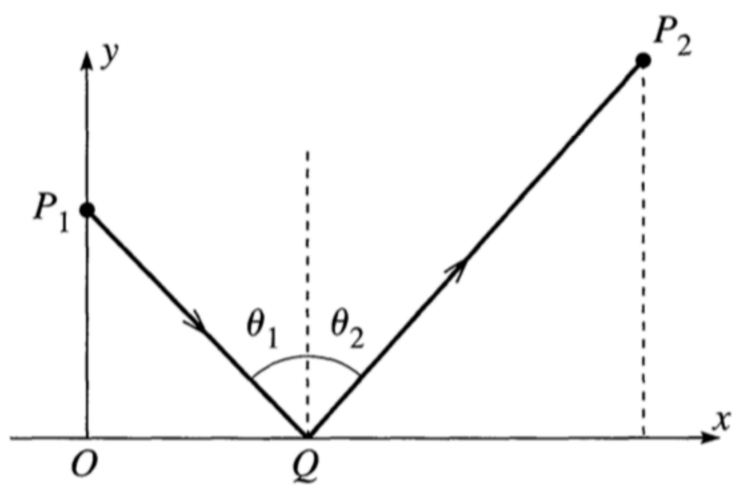
Problem Set 5

due 8 October 2018

Instructions:

1. Answer all questions below (bonus questions optional).
2. Show your work for full credit.
3. All problems are due by 11:59pm on 8 October 2018.
4. You may collaborate, but everyone must turn in their own work.

1. Consider a ray of light traveling in a vacuum from point P_1 to P_2 by way of point Q on a plane mirror, as shown below. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, $\theta_1 = \theta_2$. [*Hints:* let the mirror lie in the xz plane, and let P_1 lie on the y axis at $(0, y_1, 0)$ and P_2 in the xy plane at $(x_2, y_2, 0)$. Finally, let $Q = (x, 0, z)$. Calculate the time for light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]



2. Find the equation of the path joining the origin O to the point $P(1, 1)$ in the xy plane that makes the integral $\int_O^P (y'^2 + yy' + y^2)\, dx$ stationary.

3. In general the integrand $f(y, y', x)$ whose integral we wish to minimize depends on y, y', and x. There is considerable simplification if f happens to be independent of y, that is, $f = f(y', x)$. Prove that when this happens, the Euler-Lagrange equation reduces to the statement that $\partial f/\partial y' = \text{const}$.

Since this is a first-order differential equation for $y(x)$, while the Euler-Lagrange equation is generally second order, this is an important simplification and the result is sometimes called a *first integral* of the Euler-Lagrange equation. In Lagrangian mechanics, we'll see that this simplification arises when a component of momentum is conserved.

4. You are given a string of fixed length l with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x axis in such a way as to enclose the maximum area between the string and the x axis. Show that the required shape is a semicircle. The area enclosed is $\int y\,dx$, but show that you can rewrite this in the form $\int_O^l f\,ds$, where s denotes the distance measured along the string from O, where $f = y\sqrt{1-y'^2}$, and y' denotes dy/ds. Since f does not involve the independent variable s explicitly, you can exploit the result of problem 6.20 in your textbook.