University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 6
Solutions

1. (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal mass $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}x^2$, where x is the extension of the spring, $x = (x_1 - x_2 - l)$ and l is the spring's unstretched length. You can assume mass 1 remains to the right of mass 2 at all times. **(b)** Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1+x_2)$ (the CM position) and x (the extension), and write down the two Lagrange equations for X and x. **(c)** Solve for $X(t)$ and $x(t)$ and describe the motion.

Solution: With the given coordinates,

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \tag{1}$$

$$U = \frac{1}{2}k\left(x_1 - x_2 - l\right)^2 = \frac{1}{2}kx^2 \tag{2}$$

$$\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}kx^2 \tag{3}$$

With the new variables, we note $x_1 = X + \frac{1}{2}x + \frac{1}{2}l$ and $x_2 = X - \frac{1}{2}x - \frac{1}{2}l$. Taking the time derivatives and substituting,

$$\mathcal{L} = \frac{1}{2}m\left[\left(\dot{X} + \frac{1}{2}\dot{x}\right)^2 + \left(\dot{X} - \frac{1}{2}\dot{x}\right)^2\right] - \frac{1}{2}kx^2 = m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{4}$$

The two Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \qquad \text{or} \qquad 0 = 2m\ddot{X} \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \qquad \text{or} \qquad -kx = \frac{1}{2}m\ddot{x} \tag{6}$$

The X equation implies motion at constant velocity, $\dot{X} = \text{const} = V_o$, so $X(t) = V_o t + X_o$. In other words, the CM moves like a free particle, which makes sense since there are no external forces. The x equation is just SHM:

$$\ddot{x} = -\frac{2k}{m}x \tag{7}$$

where we know the general solution is $x(t) = A\cos(\omega t - \delta)$. The two masses oscillate in and out relative to each other with frequency $\omega = \sqrt{2k/m}$. We can view this result in several ways. On one hand, the spring compresses twice as much with both masses moving compared to only one mass moving. On the other hand, we could say it is as though the spring has an effective force constant $2k$ with both masses moving.

2. A mass m is suspended from a massless string, the other end of which is wrapped several times around a horizontal cylinder of radius R and moment of inertia I, which is free to rotate about a fixed horizontal axle. Using a suitable coordinate, set up the Lagrangian and the Lagrange equations of motion, and find the acceleration of the mass m. (The kinetic energy of a rotating cylinder is $\frac{1}{2}I\omega^2$.)

Solution: W need to account for the translational kinetic energy of the mass m moving and the rotational kinetic energy of the cylinder. If the rope is not to slip on the cylinder, then the angular velocity ω for the cylinder must match the linear velocity $\dot{x}$ of the mass m, $\dot{x} = R\omega$. Put another way, the arclength a point on the surface of the cylinder moves through in a time t must match the distance the mass has fallen, so $x = \theta R$. Taking a time derivative and noting $\dot{\theta} = \omega$, we have the same condition. Thus,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I}{R^2}\dot{x}^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 \tag{8}$$

Taking the zero of potential energy to be the starting position of the mass m, with $+x$ downward, we have $U = -mgx$, and thus

$$\mathcal{L} = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + mgx \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial x} = mg \tag{10}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m + \frac{I}{R^2}\right)\ddot{x} \tag{11}$$

$$\ddot{x} = \frac{g}{1 + \frac{I}{mR^2}} \tag{12}$$

3. Using the usual angle φ as generalized coordinate, write down the Lagrangian for a simple pendulum of length l suspended from the ceiling of an elevator that is accelerating upward with constant acceleration a. (Be careful when writing T; it is probably safest to write the bob's velocity in component form.) Find the Lagrange equation of motion and show that it is the same as that for a normal, non-accelerating pendulum, except that g has been replaced by $g + a$. In particular, the angular frequency of small oscillations is $\sqrt{(g+a)/l}$.

Solution: The elevator is a non-inertial frame (primed coordinates), so we should analyze the situation from a reference frame on the ground (unprimed coordinates). The easiest way forward is to write down the coordinates of the mass m from a cartesian reference frame based on the ground, with y upward and x to the right. With respect to the *elevator*, with the origin at the support point, the mass' position is $(x', y') = (l \sin\varphi, -l\cos\varphi)$. We add to this the position of the elevator relative to the ground, $(0, \frac{1}{2}at^2)$, and find the position of the mass as:

$$x = l\sin\varphi \tag{13}$$
$$y = \frac{1}{2}at^2 - l\cos\varphi \tag{14}$$

Taking the time derivative, and remembering the chain rule,

$$\dot{x} = v_x = l\dot{\varphi}\cos\varphi \tag{15}$$
$$\dot{y} = v_y = l\dot{\varphi}\sin\varphi + at \tag{16}$$

The height of the bob above the ground is $y = \frac{1}{2}at^2 - l\cos\varphi$, making the potential energy mgy. We can then construct the Lagrangian, with $v^2 = v_x^2 + v_y^2$:

$$\mathcal{L} = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m\left(a^2t^2 + 2atl\dot{\varphi}\sin\varphi + l^2\dot{\varphi}^2\right) - mg\left(\frac{1}{2}at^2 - l\cos\varphi\right) \tag{17}$$

The Lagrange equation is then readily obtained. First the relevant derivatives.

$$\frac{\partial\mathcal{L}}{\partial\varphi} = matl\dot{\varphi}\cos\varphi - mgl\sin\varphi \tag{18}$$
$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\dot{\varphi} + matl\sin\varphi \tag{19}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\ddot{\varphi} + matl\dot{\varphi}\cos\varphi + mal\sin\varphi \tag{20}$$

The first and third lines must be equal according to the Lagrange equation.

$$matl\dot{\varphi}\cos\varphi - mgl\sin\varphi = ml^2\ddot{\varphi} + matl\dot{\varphi}\cos\varphi + mal\sin\varphi \tag{21}$$
$$-mgl\sin\varphi = ml^2\ddot{\varphi} + mal\sin\varphi \tag{22}$$
$$-g\sin\varphi = l\ddot{\varphi} + a\sin\varphi \tag{23}$$
$$l\ddot{\varphi} = -(g+a)\sin\varphi \tag{24}$$

This is the normal equation for a pendulum except that the acceleration is $g + a$ rather than g. For small angles, where $\sin\varphi \approx \varphi$, this leads to a frequency of small oscillations $\omega = \sqrt{\frac{g+a}{l}}$.

4. We saw in example 7.3 in the text that the acceleration of an Atwood machine is $\ddot{x} = (m_1 - m_2)g/(m_1+m_2)$. It is sometimes claimed that this result is "obvious" because, it is said, the effective force on the system is $(m_1 - m_2)g$ and the effective mass is $(m_1 + m_2)$. This is not, perhaps, all that obvious, but it does emerge very naturally in the Lagrangian approach. Recall that the Lagrange equation can be thought of as

$$(\text{generalized force}) = (\text{rate of change of generalized momentum}) \tag{25}$$

Show that for the Atwood machine the generalized force is $(m_1 - m_2)g$ and the generalized momentum is $(m_1 + m_2)\dot{x}$.

Solution: We've already considered the Atwood machine in Chapter 7 (example 7.3), so we already know the Lagrangian:

$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx \tag{26}$$

Using Lagrange's equation,

$$\frac{\partial \mathcal{L}}{\partial x} = (m_1 - m_2)g = (\text{generalized force}) \tag{27}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}(m_1 + m_2)\dot{x} = (m_1 + m_2)\ddot{x} = (\text{generalized momentum}) \tag{28}$$

5. A pendulum consists of a mass m and a massless stick of length l. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos\omega t$. What is the general solution for the angle of the pendulum as a function of time?

Solution: As with problem 3, we need to choose a reference frame on the ground since the pendulum support is accelerating. Again, we write down the position with respect to the ground, with x horizontal and y vertical. With $x(t) = A\cos\omega t$ for the support, the position of the mass is

$$(x, y)_m = (x + l\sin\theta, -l\cos\theta) \tag{29}$$

Now we take the time derivative to get the velocity components:

$$\dot{x}_m = \dot{x} + l\dot{\theta}\cos\theta \tag{30}$$

$$\dot{y}_m = l\dot{\theta}\sin\theta \tag{31}$$

$$v_m^2 = \dot{x}_m^2 + \dot{y}_m^2 = \dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta = \dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2 \tag{32}$$

The potential energy with y measured from the vertical position of the support is $U = -mgy = -mgl\cos\theta$, so the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta \tag{33}$$

Clearly our generalized coordinates are x and θ. Since the Lagrangian is independent of x, the coordinate x is ignorable, and Lagrange's equation for x will only indicate that momentum is conserved along x. Applying Lagrange's equation for θ,

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \tag{34}$$

$$\frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{x}\cos\theta\right) = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta \tag{35}$$

$$l\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta \tag{36}$$

Plugging in $\ddot{x} = -\omega^2 A\cos\omega t$,

$$l\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0 \tag{37}$$

It sort of looks like a normal pendulum equation, except that we have both horizontal and vertical components of acceleration. We can apply a small angle approximation for θ:

$$\ddot{\theta} - \frac{A}{l}\omega^2\cos(\omega t) + \frac{g}{l} = 0 \tag{38}$$

$$\ddot{\theta} + \omega_o^2\theta = \frac{A}{l}\omega^2\cos(\omega t) \tag{39}$$

Here $\omega_o = \sqrt{g/l}$. This is just the equation of a driven oscillator without damping, whose solution we investigated in Ch. 5:

$$\theta(t) = \frac{(A/l)\omega^2}{\omega_o^2 - \omega^2}\cos\omega t + C\cos(\omega_o t + \varphi) \tag{40}$$

where C and φ are constants, determined by the initial conditions.

6. Consider a mass m connected to a spring of force constant k, confined to the x axis. We know the system executes simple harmonic motion with frequency $\omega = \sqrt{k/m}$ if the spring is massless. Using the Lagrangian approach, you can find the effect of the spring's mass M as follows. **(a)** Assuming the spring is uniform and stretches uniformly, show that its kinetic energy is $\frac{1}{6}M\dot{x}^2$. (As usual x is the extension of the spring from its equilibrium length.) Write down the Lagrangian for the system of cart plus spring (note $U = \frac{1}{2}kx^2$ still holds.) **(b)** Write down the Lagrange equation and show that the mass still executes SHM, but with angular frequency $\omega = \sqrt{k/(m + M/3)}$; that is, the effect of the spring's mass M is just to add $M/3$.

Solution: With one end of the spring fixed, the other end moves with the velocity of the mass $\dot{x}$, and we presume the velocity varies linearly from one end to the other. If we compress or extend the spring by some amount x, its current length is then $l+x$. If the velocity is to vary linearly over the whole length, at some position x' along the spring the velocity is

$$v(x') = \frac{x'}{l+x} v \tag{41}$$

At position x', the mass dm of some infinitesimal segment dx' would be the fractional mass of the segment (the length of the segment divided by the total length) multiplied by the mass of the spring M:

$$dm = \frac{dx'}{l+x} M \tag{42}$$

Putting the two together we can find the kinetic energy of our infinitesimal segment of the spring:

$$dT = \frac{1}{2} v(x')^2 \, dm \tag{43}$$

Now we just integrate over the length of the spring, from 0 to $l+x$. Note $v = \dot{x}$.

$$T = \int_0^{l+x} dT = \int_0^{l+x} \frac{1}{2} \frac{x'^2 v^2}{(l+x)^2} \frac{M}{l+x} \, dx' = \frac{1}{2} \frac{\dot{x}^2 M}{(l+x)^3} \int_0^{l+x} x'^2 \, dx' \tag{44}$$

$$T = \frac{1}{2} \dot{x}^2 \frac{M}{(l+x)^3} \frac{x'^3}{3} \bigg|_0^{l+x} = \frac{1}{6} M \dot{x}^2 \tag{45}$$

So the spring behaves as though 1/3 of it moves with the full velocity $\dot{x}$. The Lagrangian is now readily constructed

$$\mathcal{L} = \frac{1}{6} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \tag{46}$$

The Lagrange equations show that we do indeed have SHM at the specified frequency.

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \tag{47}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{1}{3} M \ddot{x} + m \ddot{x} = -\frac{\partial \mathcal{L}}{\partial x} \tag{48}$$

$$\implies \qquad \ddot{x} = \frac{k}{m + \frac{1}{3} M} x \tag{49}$$

$$\implies \qquad \omega = \sqrt{\frac{k}{m + \frac{1}{3} M}} \tag{50}$$