HW 7
ph301 F18

1. from lecture $\Delta v = \sqrt{\frac{2GM_s}{r_1}\left(\frac{r_2}{r_1+r_2}\right)} - \sqrt{\frac{GM_s}{r_1}}$ to crash into sun

to change orbits from r_1 to r_2

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$ $M_s = 1.99 \times 10^{30}\ kg$

$r_1 = r_{es} = 1.5 \times 10^{11}\ m$ $r_2 = r_{sun} = 6.96 \times 10^{8}\ m$

$\Rightarrow \Delta v_{sun} = -26.9\ km/s$

to escape solar system, have to escape gravity of E & S

$$\frac{1}{2}mv^2 - \frac{GM_s m}{r_{es}} - \frac{GM_s m}{r_e} = 0 \text{ to escape}$$

(1) grav from sun while on E

(2) earth grav on earth

$\Rightarrow v_{esc} = 43500\ m/s$

but $\Delta v = v - v_i = v - v_{earth} = v - \sqrt{\frac{GM_s}{r_{es}}}$

$= 43.5 - 29.7\ km/s = 13.8\ km/s$

less energy to launch out of solar system

2. for circular motion $T = \frac{1}{2} m_e \omega^2 r_e^2$, $U = -\frac{G M_s m_e}{r_e}$

gravity must provide centripetal force

$$\frac{G M_s m_e}{r_e^2} = m_e \omega^2 r_e \Rightarrow \omega^2 = \frac{G M_s}{r_e^3}$$ (also: Kepler's 3rd $T = \frac{2\pi}{\omega}$)

$$\Rightarrow T = \frac{1}{2} m_e r_e^2 \cdot \frac{G M_s}{r_e^3} = \frac{G M_s m_e}{2 r_e} = -\frac{1}{2} U$$ as per

now if $M_s \longrightarrow \frac{M_s}{2}$, T is unchanged since m_e, ω, r_e are

but U decreases by $\frac{1}{2}$

$T = T'$ i/f $U' = U/2$

and $T = -\frac{1}{2} U_{orig}$ must still be true

$$E' = T' + U' = T + \frac{1}{2} U$$

$$= -\frac{1}{2} U + \frac{1}{2} U = 0$$

energy is 0 ⇒ parabolic orbit ⇒ earth escapes

will eliminate all life rather than half; inelegant

3. a) $F = ma = m_1 \omega^2 r$ radial force, centr acc

$$\frac{G m_1 m_2}{r^2} = m_1 \omega^2 r = m_1 \left(\frac{2\pi}{\tau}\right)^2 r$$

$$\frac{G m_2}{r^2} = \frac{4\pi^2 r}{\tau^2} \qquad \tau^2 = \frac{4\pi^2 r^3}{G m_2} \qquad \tau = 2\pi \frac{r^{3/2}}{\sqrt{G m_2}}$$

note m_1 doesn't affect period if fixed

b) 2 body, $F = \mu \ddot{r}$, $\mu = \frac{m_1 m_2}{M}$

radial rel: $\mu \omega^2 r = \frac{G m_1 m_2}{r^2} = \frac{4\pi^2 \mu r}{\tau^2}$

$$\tau^2 = \frac{4\pi^2 \mu r^3}{G m_1 m_2} = \frac{4\pi^2 r^3}{G M} \qquad \tau = 2\pi \frac{r^{3/2}}{\sqrt{G M}}$$

$$\left(\frac{\mu}{m_1 m_2} = \frac{1}{M}\right) \qquad \tau = \frac{2\pi r^{3/2}}{\sqrt{G(m_1 + m_2)}}$$

now both masses det period. but if $m_2 \gg m_1$

$$\sqrt{G(m_1 + m_2)} = \sqrt{G m_2 \left(1 + \frac{m_1}{m_2}\right)} \simeq \sqrt{G m_2}$$

$m_2 \to \infty$ same as fixing m_2 - if $m_2 \gg m_1$ approx that m_2 fixed is good

c) if $m_1 = m_2$, $M = 2m_2$ $\qquad \tau = \frac{2\pi r^{3/2}}{\sqrt{2 G m_2}} = \frac{\tau_{orig}}{\sqrt{2}}$

Period reduced by factor $\frac{1}{\sqrt{2}}$ to ~ 0.71 years

4.

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2, \text{ so } \mathcal{L} = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 - \frac{1}{2}kr^2$$

$\frac{\partial \mathcal{L}}{\partial R} = 0$ $\Rightarrow$ $\frac{\partial \mathcal{L}}{\partial \dot{R}} = \text{const} = M\dot{R}$ or $\ddot{R} = 0$ $\Rightarrow$ CM moves w/ const vel

R ignorable

$$\frac{\partial \mathcal{L}}{\partial r} = -kr = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}(\mu\dot{r}) = \mu\ddot{r} \qquad \underline{\underline{\mu\ddot{r} = -kr}} \quad \text{SHM}$$

rel posn moves like 2D isotropic osc, $\omega = \sqrt{\frac{k}{\mu}}$

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5. a) If r is fixed $\frac{\partial U_{eff}}{\partial r} = 0 = \mu\ddot{r}$

$$\frac{\partial U_{eff}}{\partial r} = \frac{Gm_1m_2}{r^2} - \frac{l^2}{\mu r^3} = 0 \qquad \frac{Gm_1m_2}{r^2} = \frac{l^2}{\mu r^3}$$

$$Gm_1m_2 = \frac{l^2}{\mu r} \qquad r_0 = \frac{l^2}{\mu Gm_1m_2} = \underline{\underline{\frac{l^2}{\mu\gamma}}}$$

b) Stable if $\frac{\partial U_{eff}}{\partial r} = 0$ is concave up, or $\frac{\partial^2 U_{eff}}{\partial r^2} > 0$

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_0} = \left.\frac{-2Gm_1m_2}{r^3} + \frac{3l^2}{\mu r^4}\right|_{r_0}$$

$$= -\frac{2\gamma}{r_0^3} + \frac{3l^2}{\mu r_0^4} = -\frac{2\gamma}{r_0^3} + \frac{3\gamma\mu r_0}{\mu r_0^4} = \frac{\gamma}{r_0^3} > 0$$

$l^2 = \mu\gamma r_0$ from (a) $\Rightarrow$ stable eqm.

c) near min $U_{eff} \simeq \underset{\text{const}}{U(r_0)} + \frac{1}{2}U''(r-r_0)^2 = \text{const} + \frac{1}{2}\frac{\gamma}{r_0^3}(r-r_0)^3$

$$\Rightarrow k = \frac{\gamma}{r_0^3}, \quad \omega = \sqrt{\gamma/\mu r_0^3}$$

$$\text{or } \omega^2 = \frac{4\pi^2}{\tau^2} = \frac{\gamma}{\mu r_0^3} \qquad \tau^2 = \frac{4\pi^2\mu r_0^3}{\gamma} \quad \checkmark \text{ orbital}$$

6.

(8.54) states $\tau^2 = 4\pi^2 \frac{a^3 \mu}{\gamma} = 4\pi^2 \frac{a^3 Mm}{GMm(M+m)}$

$$\tau^2 = \frac{4\pi^2 a^3}{G(m+M)} \quad \text{so} \quad k = \frac{4\pi^2}{G(M_s + m_p)}$$

min for Jupiter $k = \frac{4\pi^2}{G(M_s + m_p)}$

min for zero (negl) mass planet $k = \frac{4\pi^2}{G}$

$$\text{frac diff} = \frac{k_{min} - k_{max}}{k_{max}}$$

$$= \frac{m_p}{M_s + m_p} \sim 10^{-3} = 0.1\%$$

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7. $h_{min} = 250\,\text{km}$, $v_{max} = 8500\,\text{m/s}$

$r_{min} = R_e + h_{min} \simeq 6650\,\text{km}$ sattelite mass negl, $\mu \simeq m$

$\ell = \text{const} = m v_{max} r_{min}$, and $c = \frac{\ell^2}{\gamma\mu} = \frac{m^2 v_{max}^2 r_{min}^2}{GM_e m \cdot m} = \frac{v_{max}^2 r_{min}^2}{GM_e}$

$$c \simeq 7960\,\text{km}$$

$$r_{min} = \frac{c}{1+\epsilon} \Rightarrow \epsilon = \frac{c - r_{min}}{r_{min}} \simeq 0.197$$

$$r_{max} = \frac{c}{1-\epsilon} \simeq 9910\,\text{km}$$

$$\Rightarrow h_{max} = r_{max} - R_e = 3510\,\text{km}$$

8.

in general for Kepler (bound, $F \sim \frac{1}{r^2}$) orbit

$$r = \frac{c}{1+\epsilon\cos\varphi} \qquad c = l^2/\gamma\mu = \text{const} \quad (G, m_1, m_2 \text{ const}, l)$$

- circular: $\epsilon = 0$, $r = c$
- parabolic, $\epsilon = 1$, $r_{par} = \dfrac{c}{1+\cos\varphi}$

  min when $\cos\varphi = 1$ @ $\varphi = 0$, so $r_{min} = \dfrac{c}{2}$