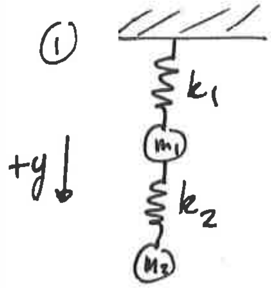
PH301 HW8
F18

(1)



Let the springs' equilibrium lengths be X_{10}, X_{20} and their extensions from equ. be x_1, x_2

$$y_1 = X_1 - X_{10} \qquad y_2 = X_2 - X_{20}$$

$$F_1 = m_1 g - k_1 x_1 + k_2(x_2 - x_1) \qquad F_2 = m_2 g - k_2(x_2 - x_1)$$

$F_1 = F_2 = 0$ in equ.

equ. $\Rightarrow$ $m_1 g = k_1 X_{10} - k_2(X_{20} - X_{10})$

$$m_2 g = k_2(X_{20} - X_{10}) \Rightarrow X_{20} - X_{10} = \frac{m_2 g}{k_2}$$

add together $m_1 g + m_2 g = k_1 X_{10}$

sub

sub

Out of equilibrium

$$F_1 = m_1 g - k_1(y_1 + X_{10}) + k_2(y_2 - y_1 + X_{20} - X_{10})$$

$$F_1 = m_1 g - k_1 y_1 - (m_1 + m_2)g + k_2(y_2 - y_1) + m_2 g$$

$$F_1 = -k_1 y_1 + k_2(y_2 - y_1) = m_1 \ddot{y}_1$$

$$F_2 = m_2 g - k_2(y_2 - y_1 + X_{20} - X_{10}) = m_2 g - k_2(y_2 - y_1) - m_2 g = -k_2(y_2 - y_1)$$

$$m_1 \ddot{y}_1 = -(k_1 + k_2) y_1 + k_2 y_2$$

$$m_2 \ddot{y}_2 = k_1 y_1 - k_2 y_2$$

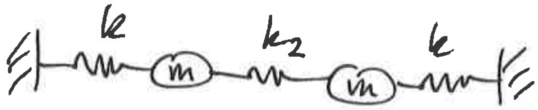
$$\Rightarrow \underline{\underline{M}}\, \ddot{\vec{y}} = -\underline{\underline{K}}\, \vec{y}$$

- as expected equ lengths and weights drop out
- they only alter equ posn not the dynamics

$$\underline{\underline{M}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \underline{\underline{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(2)



a)

$F_1 = -k x_1 + k_2(x_2 - x_1) \qquad F_2 = k_2 x_1 - (k+k_2) x_2$

$F_1 = -(k+k_2)x_1 + k_2 x_2$

$$\underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \underline{\underline{K}} = \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix} \qquad \underline{\underline{K}} - \omega^2 \underline{\underline{M}} = \begin{bmatrix} k+k_2-\omega^2 m & -k_2 \\ -k_2 & k+k_2-\omega^2 m \end{bmatrix}$$

$\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0 \Rightarrow (k+k_2-\omega^2 m)^2 - k_2^2 = 0$

$m^2\omega^4 - 2(k+k_2)m\omega^2 + (k+k_2)^2 - k_2^2 = 0$

$m^2\omega^4 - 2(k+k_2)m\omega^2 + (k^2+2kk_2) = 0$

$$\omega^2 = \frac{2(k+k_2)m \pm \sqrt{4(k+k_2)^2 m^2 - 4m^2(k^2+2kk_2)}}{2m^2}$$

$$\omega^2 = \frac{k+k_2}{m} \pm \frac{1}{m}\sqrt{k^2 + 2kk_2 + k_2^2 - k^2 - 2kk_2}$$

$$\omega^2 = \frac{k+k_2}{m} \pm \frac{1}{m}\sqrt{k_2^2} = \frac{k+k_2 \pm k_2}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}},\ \sqrt{\frac{k+2k_2}{m}}$$

with $k = k_2$, agrees with previous result

b) $\omega = \sqrt{\frac{k}{m}} \qquad \underline{\underline{K}} - \omega^2 \underline{\underline{M}} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$

Same mode as equal springs
equal amplitudes, oscillate
in unison (in phase)

$\omega = \sqrt{\frac{k+2k_2}{m}} \qquad \underline{\underline{K}} - \omega^2 \underline{\underline{M}} = \begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix}$

again same as equal springs
equal amplitudes and
exactly out of phase
due to symmetry even
though ω depends on k_2

3. we had

a)
$$\underline{\underline{M}} = \begin{bmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix} \quad \underline{\underline{K}} = \begin{bmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

$$\underline{\underline{K}} - \omega^2 \underline{\underline{M}} = \begin{bmatrix} (m_1+m_2) g L_1 - \omega^2 (m_1+m_2) L_1^2 & -\omega^2 m_2 L_1 L_2 \\ -\omega^2 m_2 L_1 L_2 & m_2 g L_2 - \omega^2 m_2 L_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} (m_1+m_2) L_1 (g-\omega^2 L_1) & -\omega^2 m_2 L_1 L_2 \\ -\omega^2 m_2 L_1 L_2 & m_2 L_2 (g-\omega^2 L_2) \end{bmatrix}$$

$$\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0 \Rightarrow \underline{m_2} \underline{L_1 L_2} (m_1+m_2)(g-\omega^2 L_1)(g-\omega^2 L_1) - \omega^4 \underline{m_2^2} \underline{L_1^2 L_2^2} = 0$$

$$L_1 L_2 m_2 \omega^4 - (m_1+m_2)(g^2 - g\omega^2(L_1+L_2) + \omega^4 L_1 L_2) = 0$$

$$m_1 L_1 L_2 \omega^4 - g(m_1+m_2)(L_1+L_2)\omega^2 + (m_1+m_2) g^2 = 0$$

$$\omega^2 = \frac{(m_1+m_2) g (L_1+L_2) \pm \sqrt{(m_1+m_2)^2 (L_1+L_2)^2 g^2 - 4 m_1 L_1 L_2 g^2 (m_1+m_2)}}{2 m_1 L_1 L_2}$$

doesn't really simplify more...

b) $m_1 = m_2 = m$, $L_1 = L_2 = L$

$$\Rightarrow \omega^2 = \frac{4mgL \pm \sqrt{4m^2 \cdot 4L^2 g^2 - 8m^2 L^2 g^2}}{2mL^2} = \frac{4gL \pm \sqrt{8L^2 g^2}}{2L^2}$$

$\omega^2 = \frac{g}{L}(2 \pm \sqrt{2})$ as before

c) $m_2 \to 0$ back to result from (a)

$$\omega^2 = \frac{m_1 g (L_1+L_2) \pm \sqrt{m_1^2 g^2 [\underbrace{(L_1+L_2)^2 - 4L_1 L_2}_{(L_1-L_2)^2}]}}{2 m_1 L_1 L_2} = \frac{g}{2L_1 L_2}\left(L_1 + L_2 \pm (L_1 - L_2)\right)$$

$\omega^2 = \frac{g}{L_2}$ a $\frac{g}{L_1}$

$\omega^2 = g/L_2$ light lower pendulum osc. @ natural freq and upper heavier one is unaffected

$\omega^2 = g/L_1$ upper heavy pendulum swings @ natural freq, unaffected by lower one

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④ $m_1 = 8m,\ m_2 = m,\ L_1 = L_2 = L$

from prev, $$\omega^2 = \frac{(m_1+m_2)g(L_1+L_2) \pm \sqrt{(m_1+m_2)^2(L_1+L_2)^2 g^2 - 4m_1L_1L_2g^2(m_1+m_2)}}{2m_1L_1L_2}$$

now $m_1+m_2 = 9m$, $L_1+L_2 = 2L$

$$\Rightarrow \omega^2 = \frac{18mgL \pm \sqrt{324m^2L^2g^2 - 4\cdot 8\cdot 9m^2L^2g^2}}{16mL^2} = \frac{mgL}{mL^2}\left(\frac{18\pm 6}{16}\right)$$

$$\omega^2 = \left\{\frac{3g}{2L},\ \frac{3g}{4L}\right\} = \left\{\frac{3}{4}\omega_0^2,\ \frac{3}{2}\omega_0^2\right\}$$

-or- $\underline{\underline{M}} = mL^2\begin{bmatrix} 9 & 1 \\ 1 & 1\end{bmatrix}$ $\underline{\underline{K}} = mL^2\omega_0^2\begin{bmatrix} 9 & 0 \\ 0 & 1\end{bmatrix}$ $\underline{\underline{K}} - \omega^2\underline{\underline{M}} = \begin{bmatrix} 9(\omega_0^2-\omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2-\omega^2)\end{bmatrix}$

$\det(\underline{\underline{K}} - \omega^2\underline{\underline{M}}) = 0$ gives same ω's

$(\underline{\underline{K}} - \omega^2\underline{\underline{M}})\vec{a} = 0$ @ ω_1: $$\begin{bmatrix} 9(\omega_0^2 - \frac{3}{4}\omega_0^2) & -\frac{3}{4}\omega_0^2 \\ -\frac{3}{4}\omega_0^2 & \omega_0^2 - \omega_0^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2\end{bmatrix} = 0$$

$$\Rightarrow \frac{9}{4}\omega_0^2 a_1 - \frac{3}{4}\omega_0^2 a_2 = 0 \quad \Rightarrow \vec{a} = A_1\begin{bmatrix} 1 \\ 3\end{bmatrix}$$
$$3a_1 = a_2$$

@ ω_2, similar proc $\Rightarrow \vec{a} = A_2\begin{bmatrix} 1 \\ -3\end{bmatrix}$

ω_1: in phase, lower has 3x amplitude

ω_2: out of phase, lower has 3x amplitude

4b) actual motion is a superposition of normal modes

w/ $\varphi_1(0)=0$, $\varphi_2(0)=\alpha$, $\dot{\varphi}_1(0)=\dot{\varphi}_2(0)=0$

$$\mathrm{Re}\left[A_1\begin{bmatrix}1\\3\end{bmatrix}+A_2\begin{bmatrix}1\\-3\end{bmatrix}\right]=\begin{bmatrix}0\\\alpha\end{bmatrix} \Rightarrow \begin{array}{l} A_1=-A_2 \\ A_1-A_2=\frac{\alpha}{3}\end{array}$$

$$\Rightarrow A_1+A_1=2A_1=\frac{\alpha}{3} \Rightarrow A_1=-A_2=\frac{\alpha}{6}$$

Since $\omega_2=\sqrt{2}\,\omega_1$, motion is not periodic (see HW4)

(5)
$$x(t)=\sum_{n=0}^{N-1} A\cos(\omega t+n\alpha)=\sum_{1} Ae^{i(\omega t+n\alpha)} = Ae^{i\omega t}\sum_1 e^{in\alpha}$$
(take real part later)

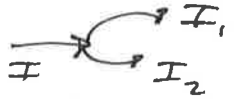
$$= Ae^{i\omega t}\sum_{n=0}^{N-1}\left(e^{i\alpha}\right)^n = Ae^{i\omega t}\,\frac{1-e^{in\alpha}}{1-e^{i\alpha}}$$
geometric series

$$= Ae^{i\omega t}\,\frac{e^{in\alpha/2}}{e^{i\alpha/2}}\cdot\frac{e^{-in\alpha/2}-e^{in\alpha/2}}{e^{-i\alpha/2}-e^{i\alpha/2}} = Ae^{i(\omega t+(\frac{n-1}{2})\alpha)}\,\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

taking real part,

$$x(t)=(NA)\,\frac{\sin\frac{n\alpha}{2}}{N\sin\frac{\alpha}{2}}\cos\left[\omega t+\left(\tfrac{n-1}{2}\right)\alpha\right]$$

equation for an antenna array or diffraction grating

(6)


$I=I_1+I_2$ $\quad P_1=I_1^2R_1$, $P_2=(I-I_1)^2R_2$

$$P_{tot}=I_1^2R_1+(I_1^2-2I_1I+I^2)R_2=I_1^2(R_1+R_2)+I^2R_2-2II_1R$$

$I=$ const

$$\frac{\partial P}{\partial I_1}=2I_1(R_1+R_2)-2IR_2 \Rightarrow I_1=\frac{R_2}{R_1+R_2},\; I_2=\frac{R_1}{R_1+R_2}$$

$$\frac{\partial^2P}{\partial I_1^2}=2R_1+2R_2-0>0,\; P_{tot} \text{ is minimum}$$