University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 10: Ch. 4.8-10
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1 Central Forces

Central forces have some aspects of 1D problems, which makes them particularly convenient to deal with. What do we mean by a central force? **A force that for any position is directed to a central point.** E.g., the sun - at any other point in the solar system one experiences a force toward the sun. If the central point is at the origin for convenience:

$$\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\hat{\mathbf{r}} \tag{1}$$

Example: Coulomb's law. In example 4.5 in the text, it is proven that the Coulomb force is *conservative*. Additionally, it is spherically symmetric (or rotationally invariant): the magnitude of $f(\mathbf{r})$ is independent of angle and the direction of $\mathbf{r}$ - it has the same value for all points at the same distance from the origin, so $f\mathbf{r} = f(r)$.

Remarkably, these two properties always come together: a central force that is conservative is automatically spherically symmetric, and a spherically symmetric central force is always conservative. We can prove this fairly easily in spherical polar coordinates.

[reviewed spherical polar coordinates quickly as an aside.]

2 Conservative and spherically symmetric, central forces

We can now prove that a central force is conservative if and only if it is spherically symmetric. We'll assume a conservative force $\mathbf{F}(\mathbf{r})$ and try to show that it must be spherically symmetric. Since $\mathbf{F}$ is conservative, we can construct it as the gradient of a scalar potential $U(\mathbf{r})$:

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r}) = -\hat{\mathbf{r}}\frac{\partial U}{\partial r} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial U}{\partial \theta} - \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi} \tag{2}$$

Since $\mathbf{F} = \mathbf{F}(\mathbf{r})$, only the radial component is non-zero, that is, $F_\theta = F_\varphi = 0$, which means

$$\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \varphi} = 0 \tag{3}$$

This means U is spherically symmetric since there is no dependence on θ or φ, so

$$\mathbf{F}(\mathbf{r}) = -\hat{\mathbf{r}}\frac{\partial U}{\partial r} \tag{4}$$

Since U is spherically symmetric, the same is true of $\partial U/\partial r$ and therefore $\mathbf{F}(\mathbf{r})$. This means that a central, conservative force is necessarily spherically symmetric. Proving the converse (that a central force which is spherically symmetric is necessarily conservative) follows follows almost exactly example 4.5 for the Coulomb force.

This is nice because $\mathbf{F}(\mathbf{r})$ is nearly as simple as a one dimensional force. It is not one dimensional since the direction still depends on φ and θ, but a central force problem can always be mapped to a related one dimensional problem.

3 Energy of two interacting particles

The interaction of two particles means that they exert equal and opposite forces on one another:

$$\mathbf{F}_{12} = (\text{on 1 by 2}) \tag{5}$$
$$\mathbf{F}_{21} = (\text{on 2 by 1}) \tag{6}$$
$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{7}$$

Here the last line follows from Newton's third law. We assume there are no external forces for now, but that the interactions do depend on the positions of the two particles:

$$\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2) \tag{8}$$

Here is the setup:

Figure 1: *Two interacting particles*

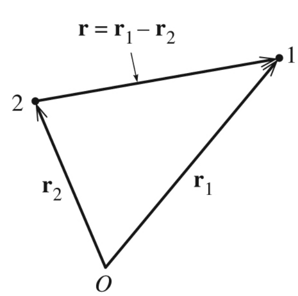

Figure 4.18
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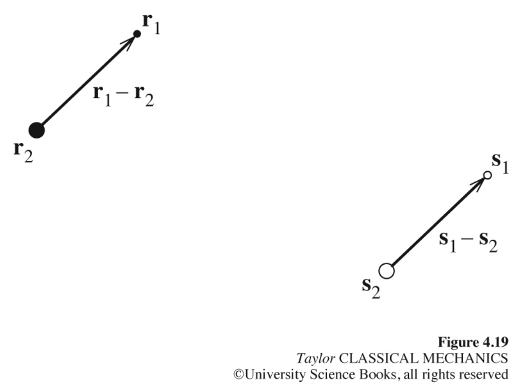
One example would be gravity:

$$\mathbf{F}(\mathbf{r}) = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = -\frac{Gm_1m_2}{r^3}\mathbf{r} = -\frac{Gm_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|^2}(\mathbf{r}_1-\mathbf{r}_2) \qquad \text{with} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{9}$$

Only the *relative* position of the two particles matters. Clearly it can't depend on the position of the origin, since that is a free choice, but it really stems from the fact that an isolated system is translationally invariant - it looks the same at any position in space. Translational invariance in fact implies momentum conservation via Noether's theorem, which we will discuss more later in the course. This is shown in the figure below - the force between the two pairs of identical particles must be the same if the only difference is a translation in space. Therefore, it must be true that $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2)$.

Figure 2: *Two sets of interacting particles*



Since we can always fix the origin at the position of particle 2, we can write $\mathbf{r}_2 = 0$ without loss of generality. Thus, $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1)$. If the force is conservative, it must satisfy

$$\boldsymbol{\nabla}_1 \times \mathbf{F}_{12} = 0 \tag{10}$$

Now we have to be a bit careful - in using the curl operator, we have to note that $\boldsymbol{\nabla}_1$ is a differential operator with respect to coordinates relative to particle 1, (x_1, y_1, z_1):

$$\boldsymbol{\nabla}_1 = \hat{\mathbf{x}}\frac{\partial}{\partial x_1} + \hat{\mathbf{y}}\frac{\partial}{\partial y_1} + \hat{\mathbf{z}}\frac{\partial}{\partial z_1} \tag{11}$$

If curl of $\mathbf{F}_{12}$ is zero, then we can write the force as the gradient of a scalar function:

$$\mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U(\mathbf{r}_1) \tag{12}$$

This gives us F_{12} when particle 2 is at the origin. If particle 2 *isn't* at the origin, we just have to shift our coordinates so $\mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2)$ since only the relative position matters. The operator $\boldsymbol{\nabla}_1$ doesn't change at all in this case since adding a constant term doesn't alter the derivatives that define it.

So what is $\mathbf{F}_{21}$ on particle 2? We know $\mathbf{F}_{12} = -\mathbf{F}_{21}$, so at the least we flip a sign. With the chain rule, you can show

$$\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U(\mathbf{r}_1 - \mathbf{r}_2) \tag{13}$$

This makes some sense: from the point of view of particle 2 the orientation has been reversed. So we can say

$$\mathbf{F}_{21} = +\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U(\mathbf{r}_1 - \mathbf{r}_2) \tag{14}$$

This means

$$\text{(force on 1)} = -\boldsymbol{\nabla}_1 U \tag{15}$$

$$\text{(force on 2)} = -\boldsymbol{\nabla}_2 U \tag{16}$$

$$\text{or} \quad \mathbf{F}_\alpha = -\boldsymbol{\nabla}_\alpha U \tag{17}$$

This is important, because it means from a *single potential* U, we get both forces - all we have to do is swap the coordinates in the $\boldsymbol{\nabla}_\alpha$ operator. This also readily generalizes to N particles - we get all interaction forces from a single potential.

Along these lines, we can show the work-energy theorem still works:

$$dT = dT_1 + dT_2 = \text{(work on 1)} + \text{(work on 2)} = d\mathbf{r}_1 \cdot \mathbf{F}_{12} + d\mathbf{r}_2 \cdot \mathbf{F}_{21} = W_{\text{tot}} \tag{18}$$

Replacing $\mathbf{F}_{21}$ with $-\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ with $-\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2)$,

$$W_{\text{tot}} = (d\mathbf{r}_1 - d\mathbf{r}_2) \cdot \mathbf{F}_{12} = d(\mathbf{r}_1 - \mathbf{r}_2) \cdot [-\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2)] \tag{19}$$

Noting that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, then the right side of the equation above is just the negative of the change in potential energy, so

$$W_{\text{tot}} = -d\mathbf{r} \cdot \boldsymbol{\nabla} U(\mathbf{r}) = -dU \tag{20}$$

The total work is the sum of the work done by F_{12} moving particle 1 through dr_1 plus the work done by F_{21} moving particle 2 through dr_2, and the total work is just $-dU$. That lets us recover conservation of mechanical energy:

$$d(T + U) = 0 \qquad \text{so} \qquad E = T + U = T_1 + T_2 + U \tag{21}$$

We always have two kinetic energy terms, one for each particle, but only a *single* potential energy for the pair. This must be the case - in order for there to be an interaction, there has to be two particles, so the potential should be a property of both together rather than each separately.

4 Energy of a multiparticle system

Let's start with 4, as shown below, and our work will easily generalize to N particles. Clearly $T = T_1 + T_2 + T_3 + T_4$, where $T_\alpha = \frac{1}{2} m_\alpha \dot{x}_\alpha^2$

Figure 3: *Four interacting particles*

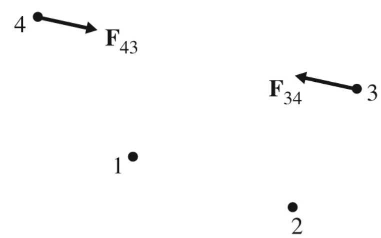


Figure 4.22
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How about the potential energy? We have internal interactions, and by superposition, they come in pairs. Since $U_{12} = U_{21}$ is the same interaction, we expect the number of interaction terms U for N particles should be

$$\text{(number of interaction terms)} = \binom{N}{2} = \frac{N!}{2!(N-2)!} \tag{22}$$

For $N = 4$, we expect 6 terms. Explicitly:

$$(1,2) \quad (1,3) \quad (1,4) \tag{23}$$
$$(2,3) \quad (2,4) \tag{24}$$
$$(3,4) \tag{25}$$

Only the relative distance between particles matters. For example, if 3 and 4 interact, then $U_{34} = U_{34}(\mathbf{r}_3 - \mathbf{r}_4)$. If the forces are conservative, then

$$\mathbf{F}_{34} = -\boldsymbol{\nabla}_3 U_{34} \tag{26}$$
$$\mathbf{F}_{43} = -\boldsymbol{\nabla}_4 U_{34} \tag{27}$$

We do this for each unique pair, $\binom{N}{2}$ times. Aside from the internal interactions, each particle can have an external force acting on it, which would depend on its position:

$$\mathbf{F}_1^{\text{ext}} = -\boldsymbol{\nabla}_\alpha U_\alpha^{\text{ext}}(\mathbf{r}_\alpha) \tag{28}$$

Adding all of the potential terms together,

$$U = U^{\text{int}} + U^{\text{ext}} = (U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}) + (U_1^{\text{ext}} + U_2^{\text{ext}} + U_3^{\text{ext}} + U_4^{\text{ext}}) \tag{29}$$

We can write this as a summation in two ways, being careful to avoid double counting and self-interactions.

$$U = \frac{1}{2}\sum_i \sum_{j\neq i} U_{ij} + \sum_i U_i^{\text{ext}} = \sum_i \sum_{j>i} U_{ij} + \sum_i U_i^{\text{ext}} \tag{30}$$

The two interaction sums (the first term in each section) are the same. The first does actually *does* double count, and then just halves the result to get the correct answer (while also noting $j \neq i$ so there are no self-interactions). The second one more explicitly avoids double counting by taking $j > i$ while summing over i, which similarly avoids counting self-interactions. We can then show that

$$-\boldsymbol{\nabla}_1 U = (\text{net force on 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \tag{31}$$

The first three terms at right represent the interaction with the other particles, the last term is the external force. In general, we can show that the net force on particle α is $-\boldsymbol{\nabla}_\alpha U$ - all forces can be derived from a *single potential*. We can then apply the work-energy theorem to each α and add the results together to show energy is conserved, just like we did for two particles: $dE = dT + dU = 0$. This is unsurprising since all the interactions we allowed are conservative, but reassuring nonetheless.