Mass on the end of a spring that obeys Hooke's law oscillates w/ SHM

$$F_x(x) = -kx \qquad x = \text{displ from equ}$$

$x=0$: no force
$x>0$: moves →, force ←
$x<0$: moves ←, force → } restoring force, equ is stable

equivalently $U(x) = \frac{1}{2}kx^2$

Consider arbitrary conservative 1D system w/ $U(x)$
Stable equ @ $x = x_0 \equiv 0$ (free choice)

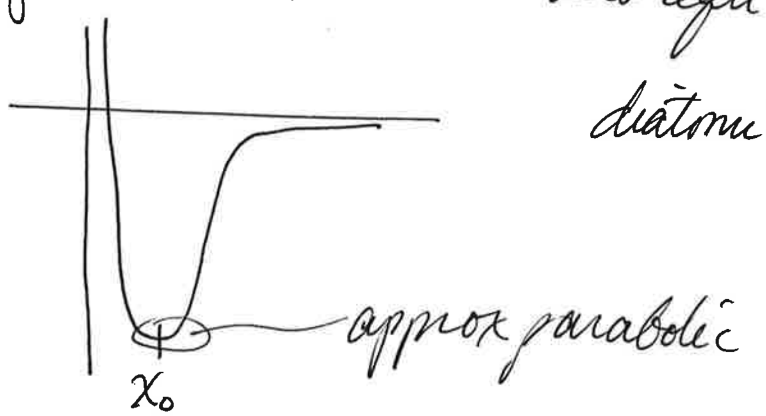
Near equ, for small displacements we can Taylor expand

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \dots$$

for small x, good approx. • because $x=0$ is equ, $U'(0) = -F(0) = 0$
• can define $U(0) = 0$

$\Rightarrow U(x) \simeq \frac{1}{2}kx^2$ w/ $k \equiv U''(0)$

- for small displ from equ, Hooke's law always valid
- if $U''(0) < 0$, $k<0$ and equ is unstable (not interested)



diatomic molecule

Cube on a cylinder ex

$U(\theta) = mg[(r+b)\cos\theta + r\theta\sin\theta]$

Small θ: $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $\sin\theta \simeq \theta$

$U(\theta) \simeq mg[(r+b)(1-\frac{1}{2}\theta^2) + r\theta^2]$

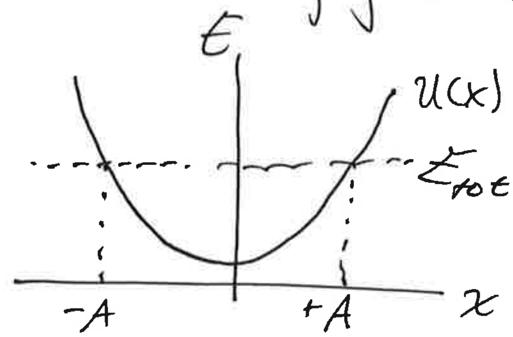
$U(\theta) = mg(r+b) + \frac{1}{2}mg(r-b)\theta^2$

let const term $= 0$ by suitable choice of $U(0) = 0$

$\Rightarrow k = mg(r-b)$, $U(\theta) - U(\theta_0) = \frac{1}{2}k\theta^2$

Stable only if $r > b$ (cylinder larger) as before!

general case



- given energy E_{tot}, particle oscillates btw $\pm A$
- A = amplitude of osc

$\boxed{\text{SHM}}$ Solve it! $F_x(x) = -kx$

$\Rightarrow m\ddot{x} = -kx$ or $\ddot{x} = -\frac{k}{m}x \equiv -\omega^2 x$ w/ $\omega^2 = \frac{k}{m}$ angular freq.

any equation in this form $\Rightarrow$ SHM

e.g. $\ddot{\varphi} = -\omega^2\varphi$ for skateboard in half pipe

variable names irrelevant

2^{nd} order, linear, homogeneous diff eqn $\Rightarrow$ 2 indep solns

Convenient choice: $x(t) = e^{i\omega t}$ and $x(t) = e^{-i\omega t}$

$\Rightarrow x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$ should work (linear combo)

- Superposn of solns, has 2 constants, so is general soln

Can recast in several ways. Using $e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$

$$\Rightarrow x(t) = (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$

$$x(t) = B_1\cos\omega t + B_2\sin\omega t \quad = \text{SHM defn}$$

w/ $B_1 \equiv C_1 + C_2 \qquad B_2 \equiv i(C_1 - C_2)$

B_1, B_2 (or C_1, C_2) relate to boundary/initial cond.

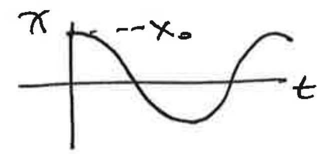
$X(0) = B_1 = x_0$ initial posn

derivative: $\dot{x}(0) = -\omega B_1 \sin\omega t + \omega B_2 \cos\omega t = \omega B_2 = v_0$

initial velocity

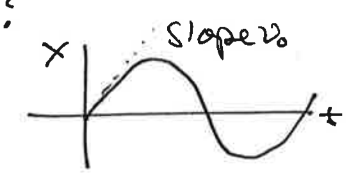
Say we release cart from $x = x_0$ at rest ($v_0 = 0$) $\Rightarrow B_2 = 0$

$$x = x_0 \cos\omega t$$



Launch from origin ($x_0 = 0$) w/ kick at $t = 0$?

$$x(t) = \frac{v_0}{\omega}\sin\omega t$$



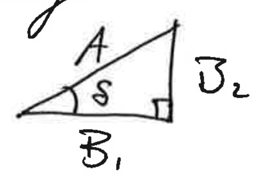
2 simplest cases. repeat? $\omega\tau = 2\pi$ -or-

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Phase-shifted cosine soln

prev is hard to visualize

let $A = \sqrt{B_1^2 + B_2^2} \Rightarrow$



can rewrite $\quad x(t) = A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right]$

$$x(t) = A\left[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right]$$

$$x(t) = A\cos(\omega t - \delta)$$

Cart oscillates w/ Ampl A. Shifted by phase δ

when $t=0$, $x(0) = A\cos\delta$

soln lags behind simple cos by phaseshift δ

$\boxed{\text{Real part of complex exp}}$

$C_1 = \frac{1}{2}(B_1 - iB_2)$ $C_2 = \frac{1}{2}(B_1 + iB_2)$ relating new coeff

note $C_2 = C_1^*$ (if $z = x+iy$, $z^* = x - iy$)

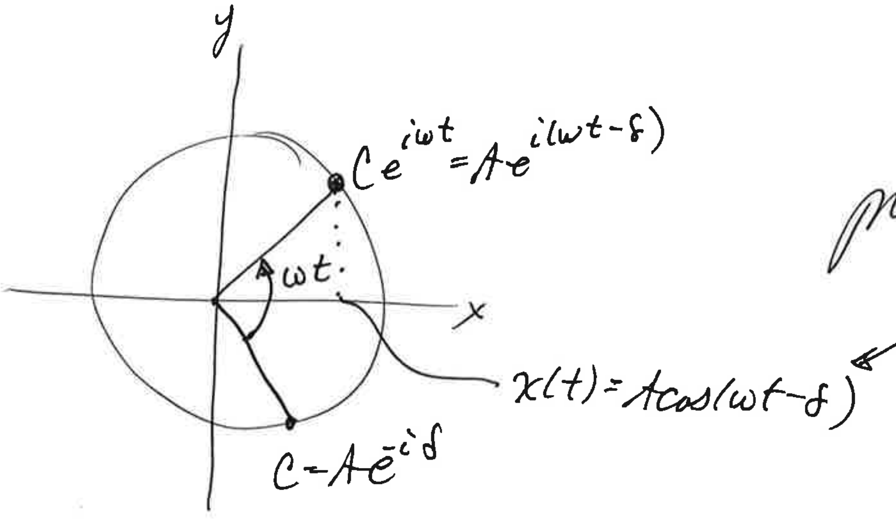
$$\Rightarrow x(t) = C_1 e^{i\omega t} + \underbrace{C_1^* e^{-i\omega t}}_{\text{complex conj of } 1^{st} \text{ term}}$$

note $z + z^* = 2\,\mathrm{Re}(z) = 2x$

$\Rightarrow x(t) = 2\,\mathrm{Re}\left[C_1 e^{i\omega t}\right]$ Let $C = 2C_1 = B_1 - iB_2 = Ae^{-i\delta}$

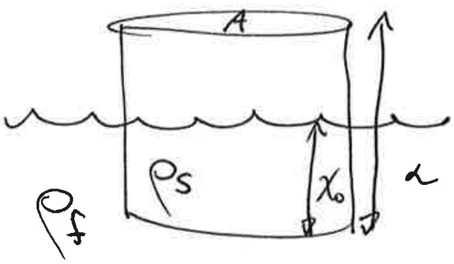
$$\Rightarrow x(t) = \mathrm{Re}\left[Ce^{i\omega t}\right] = \mathrm{Re}\left[\underbrace{Ae^{i(\omega t - \delta)}}\right]$$

moves CCW w/ angular vel ω around a circle radius A



projection along x axis is motion

Floating cylinder



buoyant: $A x_0 \rho_f g$ = wt of displaced fluid

weight: $mg = A L \rho_s g$

$$m\ddot{x} = A x_0 \rho_f g - A L \rho_s g = Ag(x_0\rho_f - L\rho_s) = 0 \text{ @ equ}$$

$$\Rightarrow x_0 = L\frac{\rho_s}{\rho_f} \qquad \rho_s < \rho_f \text{ to float!}$$

Say we're at a depth $x_0 + x$

- added buoyant force of $A x \rho_f g$

$$\Rightarrow m\ddot{x} = -A x \rho_f g \qquad \ddot{x} = -\frac{A x \rho_f g}{m}$$

from equ: $mg = A x_0 \rho_f g \Rightarrow m = A x_0 \rho_f$

$$\Rightarrow \ddot{x} = -\left(\frac{g}{x_0}\right)x \quad \text{so } \omega = \sqrt{\frac{g}{x_0}} \text{ indep } m, \rho, A!$$

same as simple pendulum w/ $l = x_0$

$d = 0.2\,m$, $\tau \simeq \frac{2\pi}{\omega} \simeq 0.9\,s$

Energy w/ $x(t) = A\cos(\omega t - \delta)$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t - \delta)$$

$$T = \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) = \tfrac{1}{2}kA^2\sin^2(\omega t - \delta)$$

w/ $\omega^2 = k/m$

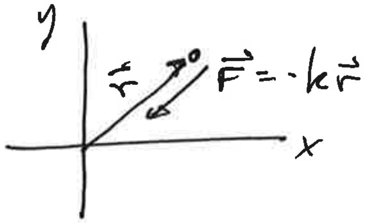
- both osc btw $0, \tfrac{1}{2}kA^2$
- perfectly out of phase U_{max} when T_{min}, vice versa

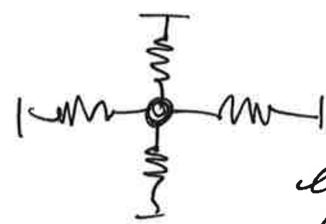
$$U + T = \tfrac{1}{2}kA^2(\sin^2 + \cos^2) = \tfrac{1}{2}kA^2 = E = \text{const}$$

as it must be for a consv. force

2D oscillators isotropic harmonic osc: $\vec{F} = -k\vec{r}$

$$F_x = -kx, \quad F_y = -ky, \quad F_z = -kz$$





equiv setup

$$\ddot{\vec{r}} = \vec{F}/m \Rightarrow \quad \ddot{x} = -\omega^2 x, \quad \ddot{y} = -\omega^2 y \qquad \omega = \sqrt{k/m}$$

Same soln

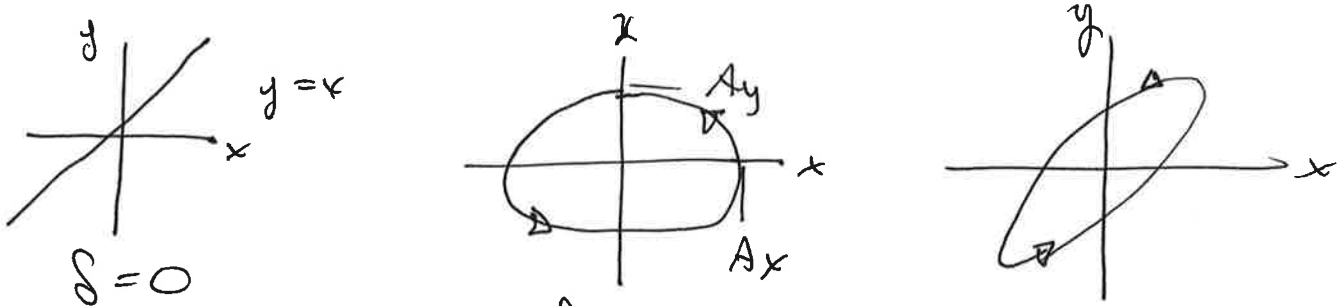
$$x(t) = A_x \cos(\omega t - \delta_x)$$
$$y(t) = A_y \cos(\omega t - \delta_y)$$

4 const b/c 2 eqns of 2nd order

Can set <u>one</u> $\delta = 0$ (choice of $t=0$)

$$x(t) = A_x \cos\omega t$$
$$y(t) = A_y \sin(\omega t - \delta)$$

A_x or $A_y = 0$? SHM along one axis
neither 0 ?



$\delta = 0$ $\delta = \pi/2$ ellipse $\delta = \pi/4$ slanted ellipse

"Lissajous curve"

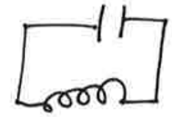
Anisotropic oscillator k diff in each dir
k_x, k_y, k_z

$\Rightarrow$ $\ddot{x} = -\omega_x^2 x$ $\omega_x = \sqrt{k_x/m}$ etc
$\ddot{y} = -\omega_y^2 y$

$\frac{\omega_x}{\omega_y}$ = rational, repeating motion - periodic
$\neq$ rational, never repeats! quasiperiodic -
x, y periodic but $\vec{r} = (x, y)$ is not

Other systems

LC circuit (series) $\ddot{q} + \frac{1}{LC} q = 0$ $L \leftrightarrow m$
$\frac{1}{C} \leftrightarrow k$



physical pend. (small θ) $\ddot{\theta} + \frac{m g l_{cm}}{I_s} \theta = 0$

- about pivot S a distance l_{cm} from center of mass
- I_s = moment of inertia about S