Damped Osc  we had $m\ddot{x}+kx=0$

now add back resistive force. as w/ air res, could be $\propto v$ or $\propto v^2$
we'll choose $\vec{f}=-b\vec{v}$ as a special case

- simple solns, instructive
- circuit analogy utility

$\Rightarrow \quad m\ddot{x}+b\dot{x}+kx=0$  cf series LRC  $I=\dot{q}(t)$

$m \rightarrow L,\ b \leftrightarrow R,\ k \leftrightarrow \frac{1}{C}$  $L\ddot{q}+R\dot{q}+\frac{1}{C}q=0$ same eqn!

rewrite $\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$  as prev, $\omega_0^2=\frac{k}{m}$ (b/c damping alters ω)

and let $2\beta=\frac{b}{m}$

$\beta$ = damping constant (convenience – mass normalized). $(\beta, \omega_0)=\frac{1}{\text{time}}$ = freq

$\Rightarrow \ddot{x}+2\beta\dot{x}+\omega_0^2x=0$  2nd order, linear, homog
$\Rightarrow$ find any 2 solns, $x_1(t)$ & $x_2(t)$
linear combo is it $C_1x_1(t)+C_2x_2(t)$
$\Rightarrow$ can guess

- function, 1st, 2nd derivs all same. guess exp.

$x(t)=e^{rt}$  $\dot{x}=re^{rt}$  $\ddot{x}=r^2e^{rt}$

$\Rightarrow r^2+2\beta r+\omega_0^2=0$. auxiliary eqn for diff eq

$\Rightarrow r_{1,2}=-\beta\pm\sqrt{\beta^2-\omega_0^2}$, $e^{r_1t}$ and $e^{r_2t}$ are 2 indep solns

So soln is $x(t)=C_1e^{r_1t}+C_2e^{r_2t}$

$$=e^{-\beta t}\left[C_1e^{\sqrt{\beta^2-\omega_0^2}\,t}+C_2e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

- Soln is not very transparent. Look @ special cases.

$\boxed{\text{Undamped}}$ we know this one! $\beta = 0$

$\Rightarrow r_1 = i\omega_1 \; r_2 = -i\omega \qquad x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$

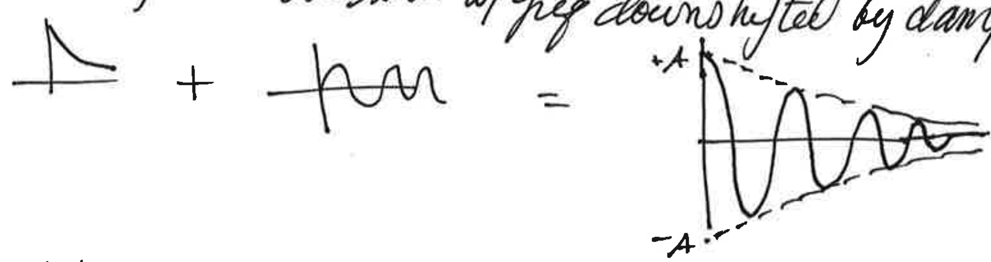
$\boxed{\text{Weak Damping}}$ small $\beta$, specifically $\beta < \omega_0$

then $\beta^2 - \omega_0^2 < 0$, so $\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} \equiv i\omega_1$

w/ $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ ($\omega_1 = \omega_0$ if $\beta = 0$) i.e. natural SHM freq

then $r_{1,2} = -\beta \pm i\omega_1$

$$\Rightarrow x(t) = \underbrace{e^{-\beta t}}_{\text{exp decay}} \underbrace{\left[C_1 e^{i\omega_1} + C_2 e^{-i\omega_1}\right]}_{\text{prev SHM osc soln w/ freq downshifted by damping}}$$




from last time, can recast as phase-shifted cos

$x(t) = e^{-\beta t} \cos(\omega_1 t - \delta)$

$\frac{1}{\beta}$ = time constant for decay! ampl reduces by factor 1/e in this time

- underdamped ($\beta < \omega_0$): $\beta$ = decay parameter
- larger $\beta$, more rapidly oscillations die out

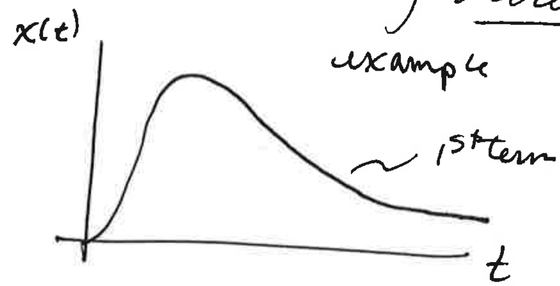
**Strong damping** $\beta > \omega_0$ "overdamping"

now $r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$ is purely real. all EXP all the time.

$$x_1(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

2 real exp functions. So damped, no osc!

- 1st term decreases more slowly since arg of exp() is smaller
  $\Rightarrow$ it characterizes long term motion
- (decay parameter) $= \beta - \sqrt{\beta^2 - \omega_0^2}$ overdamped
- unexpected: decay <u>decreases</u> as $\beta \uparrow$




- kicked from origin at $t=0$
- out toward max displ, then relaxes back to $x=0$

**Critical damping** what if $\beta = \omega_0$? edge btw over/under damping

problem: if $\beta = \omega_0$, our 2 solns are degenerate (both are same)

$$x(t) = e^{-\beta t}$$

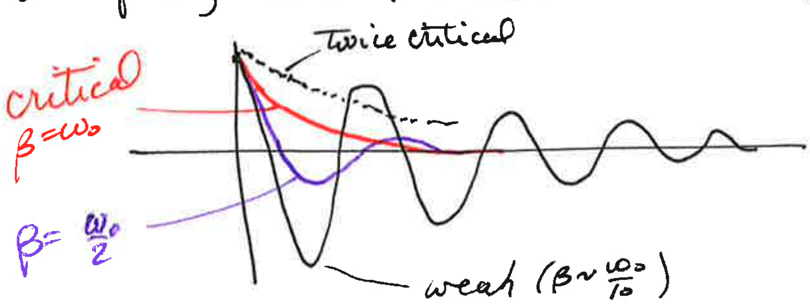
need a second guess

turns out $x(t) = t e^{-\beta t}$ works too

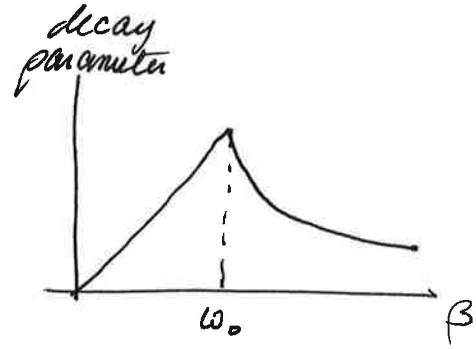
$$\Rightarrow x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

both terms decay w/ rate $\beta$
so decay of osc from both terms is (decay param) $= \beta = \omega_0$

damping turned on @ $t=0$



| damping | $\beta$ | decay parameter |
|---|---|---|
| none | $\beta=0$ | $0$ |
| under | $\beta<\omega_0$ | $\beta$ |
| critical | $\beta=\omega_0$ | $\beta$ |
| over | $\beta>\omega_0$ | $\beta-\sqrt{\beta^2-\omega_0^2}$ |




$\beta=\omega_0$ max decay - just smoothly to zero

e.g. rapidly settling meter (analog)
shock absorbers (no osc!)
door damper
cabinets... etc
PID controller

* overdamped also has no osc, but won't settle as quickly

Circuit analogy $\beta=\frac{R}{2L}$, $\omega_0=\sqrt{\frac{1}{LC}}$. critical when $R=2\sqrt{\frac{L}{C}}$

(problem: temperature dep of resistance ... not too close)

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Driven damped osc free-swinging vs pushed?

Say net force $F(t)$ drives the system (eg. sinusoidal)

$$m\ddot{x}+b\dot{x}+kx=F(t) \Leftrightarrow L\ddot{q}+R\dot{q}+\frac{1}{C}q=V(t)$$

let $f(t)=\frac{F(t)}{m}$, then as before $\div$ by $m$

$$\ddot{x}+2\beta\dot{x}+\omega_0^2x=f(t)$$

Can't solve for just any $f(t)$, but a sinusoidal drive is both interesting and practical

253: rad from accel. charges

"problem": previously solved $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$
can add this soln to new eqn * since its zero! *

let $D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$

we solved $DX_h = 0$ homogenous soln

now we want $DX_p = f$ particular soln

we know $X_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Since eqns are linear, $D(X_p + X_h) = DX_p + DX_h = f + 0 = f$

$\Rightarrow$ find $X_p$ for given $f$ and we're done

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Sinusoidal drive, e.g. $f(t) = f_0 \cos\omega t$

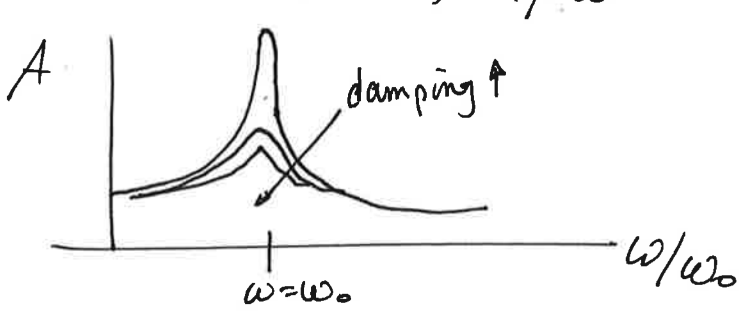
for simplicity, let $f = f_0 e^{i\omega t}$ - fake complex soln

trial soln: response osc @ same freq $z = Ce^{i\omega t} = x + iy$
(take real part to get $x$ @ end)

$\Rightarrow$ $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} = (-\omega^2 + 2i\beta\omega + \omega_0^2) Ce^{i\omega t}$

this works if $C = \dfrac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = \underbrace{Ae^{-i\delta}}_{\text{last time - any complex \#}}$

or $A^2 = \dfrac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$ hugely peaked @ $\omega = \omega_0$




• why the sky is blue ... $\omega_0 = UV$ for $N_2, O_2$
scattering stronger for higher freq
meaning blue ...

• huge amplitude @ $\omega = \omega_0$ - resonance! pushing exactly right

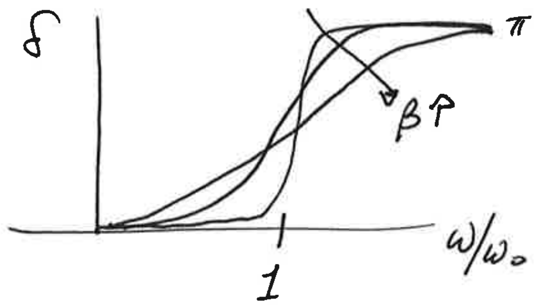
noting $C = Ae^{-i\delta}$ can find $\delta$ too $Ae^{i\delta} = \dfrac{f_0}{\omega_0^2-\omega^2+2\beta i\omega}$

$\Rightarrow$ $f_0 e^{i\delta} = A(\omega_0^2-\omega^2+2i\beta\omega)$

• $f_0, A$ are real

• phase is same as term in brackets

$$\delta = \arctan\left(\frac{\text{img}}{\text{real}}\right) = \arctan\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$




at resonance, goes from $0 \rightarrow \pi$

so $z = Ce^{i\omega t} = Ae^{i(\omega t-\delta)}$

real part $x(t) = A\cos(\omega t-\delta)$ $A, \delta$ as above

this is just particular soln...

full: $x(t) = A\cos(\omega t-\delta) + \underbrace{C_1e^{r_1t} + C_2e^{r_2t}}$

die out exponentially - transients

- our particular soln is steady state
- can specify after several $\beta$'s / decay times transients should be negl

$\boxed{\text{KEYs}}$ restoring ($kx$) and damping ($b\dot{x}$) forces are linear

diff eq is linear!

very special and unusual case

• real cases can be amazingly diff - nonlinear dynamics

• no hope there w/o understanding linear system

Say its weakly damped

$$x(t) = \underbrace{A\cos(\omega t - \delta)}_{\text{driven}} + \underbrace{A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr})}_{\text{transient}}$$

$A, \delta$ NOT arbitrary
det by $\omega, \omega_0, \beta, f_0$
i.e. system + drive

transient, and consts $A_{tr}, \delta_{tr}$
exp decay $\Rightarrow$ irrelevant for long term
dynamics (after several $\beta$)

e.g.