University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 13: Ch. 5.6

21 Sept 2018

1 Resonance

Recall from last time our solution for the damped oscillator

$$x(t) = A\cos(\omega t + \delta) \tag{1}$$
$$A^2 = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{2}$$

As expected, A^2 is proportional to the energy of the oscillator, and the amplitude A is proportional to the drive strength, $A \propto f_o$.

The most interesting cases are when β is small, meaning the second term in the denominator for A^2 is small.

- if ω and ω_o are very different, the first term in the denominator is large and the amplitude is small
- if ω is close to ω_o, both terms in the denominator are small and the amplitude is large
- dramatic change as you vary ω or ω_o

Here's a plot for small β.

Figure 1: *Amplitude squared of a driven oscillator as a function of natural frequency ω_o with the drive frequency ω fixed.*

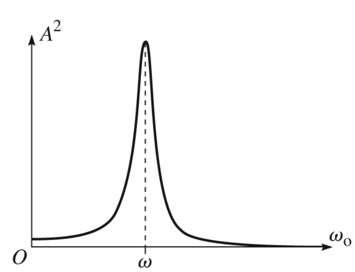

Figure 5.16
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Left alone, the system vibrates at ω_o. Trying to force it at a drive frequency ω? Responds well only if $\omega \approx \omega_o$ - this is resonance! For example, a radio. Adjusting the dial, you are varying the resonant frequency of a tuned LCR circuit. When its ω_o matches the broadcast frequency ω, you get a large response. This produces a voltage in your radio circuit (which is then demodulated, etc.). Only when $\omega \approx \omega_o$ do you get an appreciable signal excited.

It does depend on whether we vary ω or ω_o, since the denominator of A^2 is $\left(\omega_o^2 - \omega^2\right)^2 + 4\beta^2\omega^2$.

- vary ω_o with ω fixed (radio)? Denominator is minimum when $\omega = \omega_o$, the first term is zero
- very ω with ω_o fixed? Max when $\omega = \sqrt{\omega_o^2 - 2\beta^2} \equiv \omega_2$ because *both* terms have ω, while only the first has ω_o
- for $\beta \ll \omega_o$, the difference is small

Now we have several frequencies to keep track of:

$$\omega_o = \sqrt{\frac{k}{m}} \qquad \text{natural frequency of system} \tag{3}$$

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} \qquad \text{frequency of damped oscillations} \tag{4}$$

$$\omega \qquad \text{frequency of driving force} \tag{5}$$

$$\omega_2 = \sqrt{\omega_o^2 - 2\beta^2} \qquad \text{value of } \omega \text{ at which response is maximum} \tag{6}$$

At the maximum, $\omega = \omega_o$ and $A_{\max} = \frac{f_o}{2\beta\omega_o}$. For maximum response you want to minimize damping, adjust the drive frequency to match ω_o, and lower frequencies are better if all else is the same.

2 Width of resonance - Q factor

Clearly, as we increase β the width of the resonance increases. The typical figure of merit is the Full Width at Half Maximum (FWHM), meaning the breadth of the peak at half its maximum intensity. In the homework you'll show that FWHM$\approx 2\beta$. The *sharpness* of the resonance is the ratio of this width to the resonance frequency, and is known as the "quality factor" or Q factor:

$$Q = \frac{\omega_o}{2\beta} = 2\pi \frac{\text{energy stored}}{\text{dissipation per cycle}} \tag{7}$$

A large Q means a narrow resonance, which is what you would want for something like a clock. Here are some typical Q values for common oscillators.
Another view of the Q factor is that oscillations die out with a decay time of $\approx 1/\beta$, but the period of oscillation is $2\pi/\omega_o$, so

$$Q = \pi \frac{\text{decay time}}{\textit{period}} \tag{8}$$

Figure 2: *Amplitude for driven oscillations for various β.*

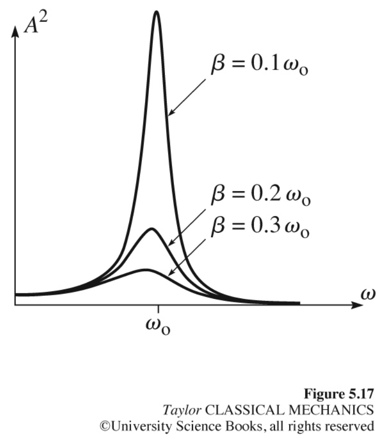
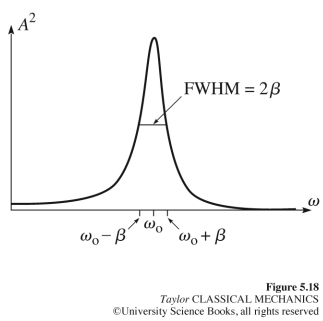


Figure 3: *FWHM*



oscillator	Q
pendulum	100
quartz crystal oscillator	10^4
precision LCR	10^4-10^6
atom	10^8
atomic clock	10^{11}

In other words, it is a measure of how many oscillations per decay time one gets, which relates to how well you could measure time or frequency with your oscillator. For an oscillating atom (e.g., Na in a discharge lamp) the period is about 10^{-15} s, but the decay time is around 10^{-8} s, so gets around ten million oscillations before the amplitude has decayed by a factor of $1/e \approx 0.37$.

3 Phase at resonance

The phase we found to be

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right) \tag{9}$$

What happens if you vary ω from well below resonance (for small β)? For $\omega \ll \omega_o$, δ is small and the oscillations are in step with the drive. As ω approaches ω_o, δ slowly increases. At $\omega = \omega_o$, $\delta = \pi/2$, and the oscillations are 90° behind the drive! When $\omega > \omega_o$, the argument of the arctan is negative and δ increases beyond $\pi/2$ toward π.

Figure 4: *Phase vs drive frequency for two values of β.*

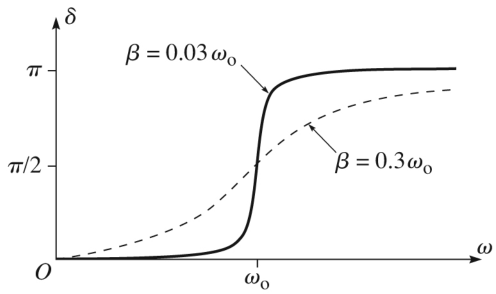


Figure 5.19
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