PH301 L14
Ch6

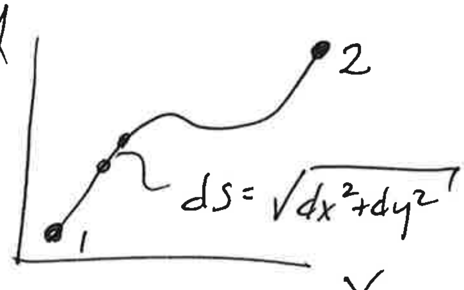
Sometimes we need non-Cartesian coord
1) system has obvious symmetry - spherical, cyl.
2) object is constrained - e.g. moving of sphere
choosing spherical reduces dimensionality
in an obvious way

But... formulas in non-Cartesian are lame & complicated
why tho? phys is same
need to reformulate! Lagrangian formulism

To understand how it works: variational principle
optimization: actual path must be optimal in some
sense! this should be coord free, i.e. coord arbitrary

6.1 Optimization examples

Shortest path btw 2 points - 2 points in a plane. Answer clear!
but derive?



$1 = (x_1, y_1)$

$2 = (x_2, y_2)$

Shortest path? $y(x)$

Segment $ds = \sqrt{dx^2+dy^2}$. note $dy = \frac{dy}{dx}dx = y'(x)dx$

so $ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'(x)^2}\,dx$

length of path is then

$$\mathcal{L} = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y'(x)^2}\, dx$$ the unknown is the function $y(x)$ minimizing $\int$ for $\mathcal{L}$!

prev: unknown is x such that $f(x) = \min$. more complex!

now imagine space isn't flat - GR [plug PHS23]

Fermat's principle — path light will follow btw 2 points

in medium $v = \frac{c}{n}$, so dist is $\frac{ds}{v}$

$$(\text{time of travel}) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\, ds$$ so how does n vary along path?

if n is const, reduces to prev problem of shortest path btw 2 points

if $n = n(x,y)$... want to minimize

$$\int_1^2 n(x,y)\, ds = \int_1^2 n(x,y)\sqrt{1+y'(x)^2}\, dx$$

very similar to prev, but added dependence on coordinates

Sometimes want max, sometimes min

Usually then we have some $f(x)$ and find $\frac{df}{dx} = 0$

doesn't guarantee min or max though!

- max
- min
- saddle

} really gives us stationary points @ x_0 s.t. $f'(x_0) = 0$

Similar to what we'll do - what are paths that make $\int$ stationary?

have to sep check if min, max, or neither

$\Rightarrow$ Calculus of variations

$\boxed{\text{Euler-Lagrange eqn}}$

general variational problem: integral of form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx$$

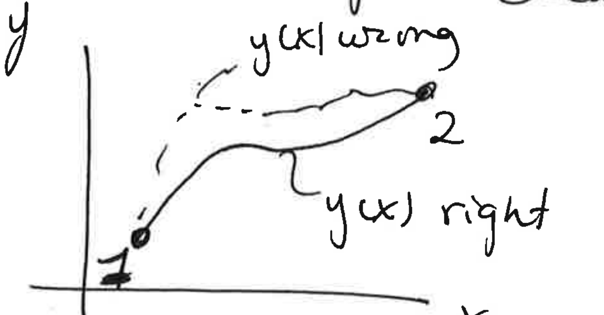
$y(x)$ is yet-unknown curve connecting (x_1, y_1) to (x_2, y_2), i.e.

$$y(x_1) = y_1 \qquad y(x_2) = y_2$$

Among all possible paths, want the one that makes S stationary

though f is a fn of y, y', x, path $y(x)$ relates the 3

$\Rightarrow$ integrand really just a function of x



if correct path is $y(x)$, S is less than along any neighboring curve e.g wrong path

wrong path can be $Y(x) = \underbrace{y(x)}_{\text{right}} + \underbrace{\eta(x)}_{\text{error}}$ has greater integral than $y(x)$ alone

all paths pass thru 1, 2, so we require

$$\eta(x_1) = \eta(x_2) = 0$$

still ∞ choices for η

The integral S along the wrong curve $Y(x)$ is always larger, no matter how close $y(x)$ & $Y(x)$ are. To express, write

$$Y(x) = y(x) + \alpha\eta(x)$$

Now integral S depends on parameter α. Correct curve has $\alpha = 0$. Requirement is then that $S(\alpha)$ is a minimum for $\alpha = 0$. Now just like calculus — a function $S(\alpha)$ has a min @ $\alpha = 0$. So check that $\frac{dS}{d\alpha} = 0$ at $\alpha = 0$

$$S(\alpha) = \int_{x_1}^{x_2} f(Y, Y', x)\,dx = \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x)\,dx$$

Want $\frac{dS}{d\alpha}$, will need $\frac{\partial f}{\partial \alpha}$. Chain rule: $\underline{x \text{ isn't } \alpha, y \;\&\; y' \text{ aren't}}$

$$\frac{\partial f(y + \alpha\eta, y' + \alpha\eta', x)}{\partial \alpha} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}$$

then for $dS/d\alpha$

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\,dx = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right) dx = 0$$

true for any η that is zero at x_1 & x_2

look @ second term ... can integrate by parts to get η back

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\,dx = \left[\eta(x) \frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

but $\eta = 0$ @ endpoints!

Thus, $\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'} dx = -\int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$

So: $\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$

has to work for <u>any</u> η, so ( ) = 0 !

$$\boxed{\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad \text{Euler-Lagrange}}$$

What will show: if $f = \mathcal{L} = T - U$, then path is det by $\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$ correct path det by energy optimization!

<u>Application: Shortest path b/two 2 points</u> in flatspace $\mathcal{L} = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

Standard form, w/ $f = f(y, y', x) = \sqrt{1+y'^2}$

to use Euler-Lagrange, need deriv of f

$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \Rightarrow \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ or $\frac{\partial f}{\partial y'} = \text{const}$

but $\frac{\partial f}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}} = \text{const} = c$

$y'^2 = c^2(1+y'^2) \Rightarrow y'^2(1-c^2) = c^2$ or $y'^2 = \text{const}$

implies $y'(x) = \text{const} = m$. So $y'(x) = m$,

$\Rightarrow y(x) = mx + b$ ✓