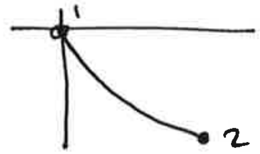
Ph301
LS ch6

Euler-Lagrange

have an "action" integral to extremize / make stationary

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx \Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Example: Brachistochrone (shortest time) (Brachistos chronos)



what path gets from 1→2 in the least time?

$$t_{12} = \int_1^2 \frac{ds}{v}$$ as before, $ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}\, dy$

where $x'(y) = \frac{dx}{dy}$

we know $v = \sqrt{2gy}$ starting from rest from cons. energy

$$\Rightarrow t_{12} = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'(y)^2 + 1}}{\sqrt{y}}\, dy$$ setting $y_1 = 0$ as choice of origin

to minimize? problem we solved last time, with x : y reversed

$$f(x, x', y) = \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}$$

So apply Euler-Lagrange w/ x, y swapped

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$$ f is indep x, so this say $\frac{\partial f}{\partial x'} = \text{Const}$

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y}\sqrt{1 + x'^2}} = \text{const}$$ squared for convenience

$$\frac{x'^2}{y(1 + x'^2)} = \text{const} = \frac{1}{2a} \leftarrow$$ for future clarity

$$\Rightarrow x' = \sqrt{\frac{y}{2a-y}} = \frac{dy}{dx}, \text{ so } x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

Let $y = a(1-\cos\theta)$ $dy = a\sin\theta\, d\theta$

$$\Rightarrow x = \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\, a\sin\theta\, d\theta = a\int \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1+\cos\theta)}}\sin\theta\, d\theta$$

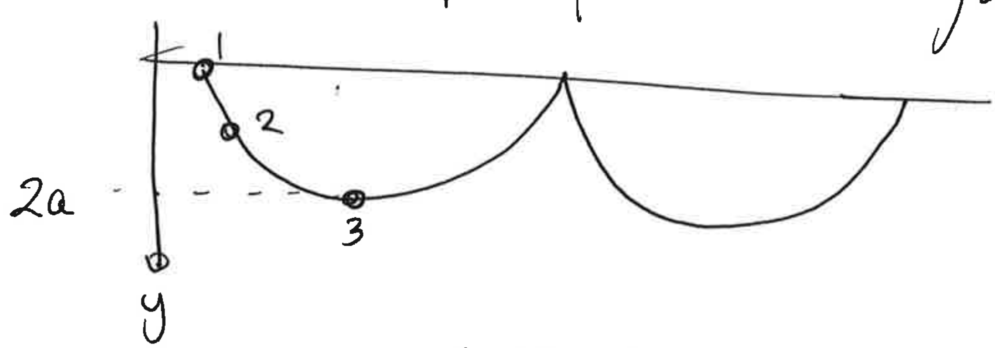
$$= a\int \sqrt{\frac{(1-\cos\theta)^2}{\underbrace{1-\cos^2\theta}_{\sin^2\theta}}}\sin\theta = a\int 1-\cos\theta\, d\theta = a(\theta - \sin\theta)$$

So

$$\left.\begin{aligned} x(\theta) &= a(\theta-\sin\theta) \\ y(\theta) &= a(1-\cos\theta) \end{aligned}\right\} \text{parametric!}$$

inverted cycloid

- cycloid = path point on rim of wheel radius a follows (if running upside down along x)



- remarkable: time to 3 is the same no matter where you start!

- isochronous exactly – perfect SHM for any displacement

More than 2 variables

- did shortest path for $y(x)$, but not all paths can be written this way!
- do it parametrically to be sure. advantage: same as $x(t)$, $y(t)$

$$\text{let } x = x(u)\,,\quad y = y(u)$$

u = anything that can parameterize curve

[to get back $y(x)$, let $u = x$]

now. $ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(u)^2 + y'(u)^2}$ $\quad x'(u) = \frac{dx}{du} \quad y'(u) = \frac{dy}{du}$

so path length is

$$L = \int_{u_1}^{u_2} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

now two unknown functions!

now we have to extremize integral of form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

between 2 fixed points $[x(u_1), y(u_1)]$ & $[x(u_2), y(u_2)]$

really, though, proc is basically as before since the x and y are indep. Just get 2 Euler-Lagrange eqns

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'}\,,\qquad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}$$

④

Shortest path again

$$f(x,x',y,y') = \sqrt{x'^2+y'^2}$$

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are zero (f indep x, y)

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2}} = \text{const} = C_1$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2}} = \text{const} = C_2$$

divide $\frac{dy}{dx} = \frac{y'}{x'} = \frac{C_2}{C_1} = m$

i.e. $y = mx+b$ ✓

Lagrangian mech: indep variable is t

dep. variables are generalized coordinates that specify the configuration of a system

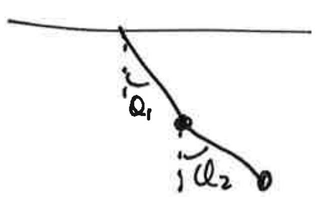
- number of coords depends on system

particle in 3D, 3 $\quad$ xyz or $r\theta\varphi$ or whatever

N particles in 3D, 3N

pendulum: 1 = θ

"coordinate" more abstract - how to specify config.



double pendulum: "coord" are θ_1, θ_2

"generalized coord" $q_1, q_2, q_3 \ldots$

momentum $\dot{q}_1, \dot{q}_2 \ldots$

$\mathcal{L}$

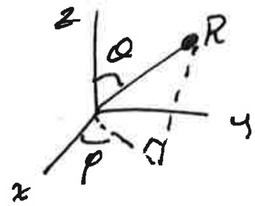
general problem: construct Lagrangian of system $(T-U)$

assert actual path ~~~~ gives stationary action

$$S = \int \mathcal{L}(q_1 \dot{q}_1, q_2, \dot{q}_2, t)\, dt \Rightarrow \frac{\partial \mathcal{L}}{\partial q_n} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$$

6.1 Shortest path on curved surface = geodesic

Sphere?



need element of length

$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

on a sphere, $dr = 0$, so $ds^2 = d\vec{s}\cdot d\vec{s} = (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2$

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) = (r\,d\theta)^2\left(1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right)$$

$$\Rightarrow \quad ds = r\,d\theta\left[1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]^{1/2}$$ assuming $\varphi = \varphi(\theta)$

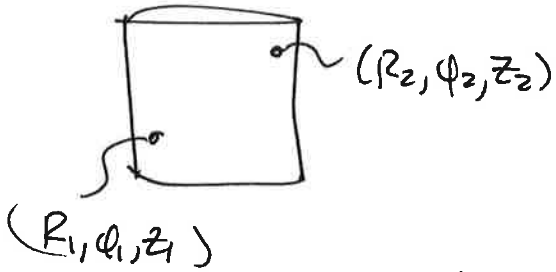
$$L = R\int_{\theta_1}^{\theta_2} d\theta\,\sqrt{1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2} = R\int \underbrace{\sqrt{1 + \sin^2\theta\,\varphi'^2}}_{f}\,d\theta$$

fixed R

now we use $\dfrac{\partial f}{\partial \varphi} - \dfrac{d}{d\theta}\dfrac{\partial f}{\partial \varphi'} = 0$ just relabel coord!

can show shortest dist = great sphere

6.7 Squirrels fastest way up cylinder? [unwrap: straight line. what's shape when wrapped]



need $\varphi(z)$ at fixed R

$$ds^2 = (R\,d\varphi)^2 + (dz)^2 = (R^2\varphi'^2 + 1)(dz)^2$$

so $$L = \int_{\varphi_1 z_1}^{\varphi_2 z_2} ds = \int_{\varphi_1 z_1}^{\varphi_2 z_2} \sqrt{1+R^2\varphi'^2}\,dz = \int f[\varphi(z), \varphi', z]\,dz$$

so $$\frac{\partial f}{\partial \varphi} - \frac{d}{dz}\frac{\partial f}{\partial \varphi'} = 0 \qquad f = \sqrt{1+R^2\varphi'^2}$$

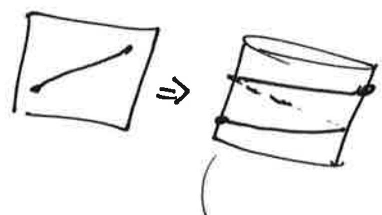
$$\frac{\partial f}{\partial \varphi} = 0 \qquad \underbrace{\frac{\partial f}{\partial \varphi'} = \frac{2R^2\varphi' \cdot \frac{1}{2}}{\sqrt{1+R^2\varphi'^2}}}_{\text{indep. } z} \qquad \text{so } \frac{d}{dz}\frac{\partial f}{\partial \varphi'} = 0$$

so $$\frac{\partial f}{\partial \varphi'} = \text{const} = C = \frac{R^2\varphi'}{\sqrt{1+R^2\varphi'^2}} \Rightarrow C^2(1+R^2\varphi'^2) = R^4\varphi'^2$$

$$C^2 + C^2R^2\varphi'^2 = R^4\varphi'^2 \qquad \varphi'^2\underbrace{(R^4 - C^2R^2)}_{\text{both const!}} = C^2$$

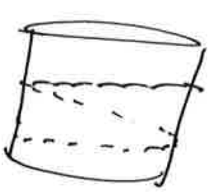
$$\Rightarrow \varphi' = \frac{d\varphi}{dz} = \text{const}$$

so $\boxed{\varphi = az + b}$

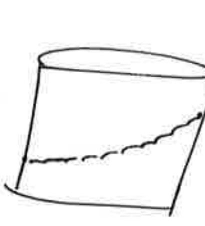


- choose a, b so curve goes through desired points
- it is a spiral!

NOT unique



vs



how many rot?

shortest when $\varphi < \pi$ in general. if φ_1, φ_2 differ by π, 2 shortest