ph301
216 Ch 7

Particle unconstrained in 3D, subject to conservative force $\vec{F}(\vec{r})$. Know KE, PE:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$U = U(\vec{r}) = U(x,y,z)$$

The Lagrangian is their difference

$$T - U = \mathcal{L}$$

- not the same as total energy. Why KE−PE?
  no simple answer - it works! try to get more PE w/ least amount of "extra" KE
- $\mathcal{L}$ depends on posn (x,y,z) and velocity $(\dot{x},\dot{y},\dot{z})$

$$\mathcal{L} = \mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z})$$

- Consider 2 derivatives, noting $U = U(x,y,z)$ and $T = T(\dot{x},\dot{y},\dot{z})$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$$

Since $F_x = \dot{p}_x$, we have $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

Same for other 2 coord y, z

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

Same form as Euler-Lagrange eqn!
The integral in question is the action S

- the actual path a particle follows from $1 \to 2$ in a time interval $t_1 \to t_2$ is such that the <u>action integral</u>

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$ is stationary. $\boxed{\text{Hamilton's principle}}$

- proved it in Cartesian coord, true for a huge class of systems and almost any choice of coordinates
- 3 equivalent statements were proven:
  1) path of particle determined by Newton's 3rd law $\vec{F} = m\vec{a}$
  2) path determined by 3 Lagrange eqns at least in Cartesian coord
     $$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$
  3) path is det by Hamilton's principle

Instead of cartesian coord, $\vec{r} = (x, y, z)$ suppose we chose some other system e.g. (r, θ, φ) or (ρ, φ, z) or any other set of "generalized coord" q_1, q_2, q_3 with the property that each posn $\vec{r}$ specifies unique (q_1, q_2, q_3) so:

$$\left.\begin{aligned} q_i &= q_i(\vec{r}) \quad i = 1, 2, 3 \\ \vec{r} &= \vec{r}(q_1, q_2, q_3) \end{aligned}\right\} \text{q_i specify $\vec{r}$ uniquely and vice versa}$$

then we can replace $(x\,y\,z)$ and $(\dot{x}\,\dot{y}\,\dot{z})$ w/ $(q_1 q_2 q_3)$ & $(\dot{q}_1 \dot{q}_2 \dot{q}_3)$

so $\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r}) = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$

$$\Rightarrow \quad S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\, dt$$

a integral is unchanged by change of variables

$\Rightarrow$ S still stationary for correct path w/ new coord

$\Rightarrow$ still satisfy Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1, 2, 3$$

- Same form w/ generalized coord! any choice of coord that uniquely specify $\vec{r}$ are fine!
- Caveat: relied on $F_x = \dot{p}_x$, true only if frame where $\mathcal{L} = T - U$ is inertial. that is: do not use accelerating ref frames
- easy to generalize to systems of many particles

$\boxed{\text{1 particle}}$ (2D cons force)

$$\mathcal{L} = \mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x,y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial U}{\partial y} = F_y$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial T}{\partial \dot{y}} = m\dot{y}$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Leftrightarrow F_x = m\ddot{x}, \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \Leftrightarrow F_y = m\ddot{y} \Leftrightarrow \vec{F} = m\vec{a}$$

w/ generalized coord q_i, $\frac{\partial \mathcal{L}}{\partial q_i}$ is not necessarily a force component but plays a very similar role. Similarly, $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ acts like momentum

$\frac{\partial \mathcal{L}}{\partial x} = F_x \Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = i^{th}$ component of generalized force

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = p_x \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = i^{th}$ component of generalized momentum

$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ gen force = rate of chg of gen mom.

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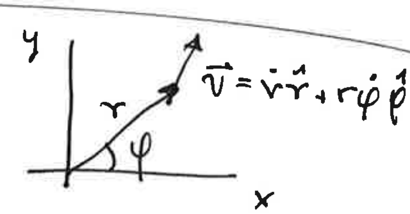
1 particle in 2D polars $\quad v_r = \dot{r} \quad v_\varphi = r\dot{\varphi}$



$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$\Rightarrow \mathcal{L} = \mathcal{L}(r,\varphi,\dot{r},\dot{\varphi}) = T - U = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r,\varphi)$$

r eqn

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\varphi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$ but $-\frac{\partial U}{\partial r} = F_r$, so

$$F_r = m\ddot{r} - mr\dot{\varphi}^2 = m(\ddot{r} - \underbrace{r\dot{\varphi}^2}_{\text{centripetal accel}})$$

or $F_r = ma_r$!

φ eqn

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \Rightarrow = -\frac{\partial U}{\partial \varphi} = \frac{d}{dt}(mr^2\dot{\varphi}) = \frac{dL}{dt}!$$

to interpret, need to know force components in polar

$$\nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \varphi}\hat{\varphi}$$

φ component of force is then $F_\varphi = -\frac{1}{r}\frac{\partial U}{\partial \varphi}$

thus $-\frac{\partial U}{\partial \varphi} = rF_\varphi$ = torque Γ on particle about origin

also: $mr^2\dot{\varphi} = mr^2\omega$ = angular momentum L about O

so we get $\Gamma = \frac{dL}{dt}$ from φ eqn!

generalized coord gives us something that plays a similar role as force - torque in this case

- φ component of gen force $= \frac{\partial \mathcal{L}}{\partial \varphi} = \Gamma$ (torque)

  φ comp of gen momentum $= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = L$ (angular momentum)
- "natural" choice of coord $\Rightarrow$ "natural" interesting quantities
- if $\Gamma = 0$, $\frac{dL}{dt} = 0$, L is conserved
- in general, if $\underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\text{gen force } i \text{ comp}} = 0$, gen momentum i comp $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is consv.

i.e. zero gen force $\Leftrightarrow$ conservation law - Noether's theorem

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Several Unconstrained Particles

$$\mathcal{L}(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) = \tfrac{1}{2}m\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m\dot{\vec{r}}_2^{\,2} - U(\vec{r}_1, \vec{r}_2)$$

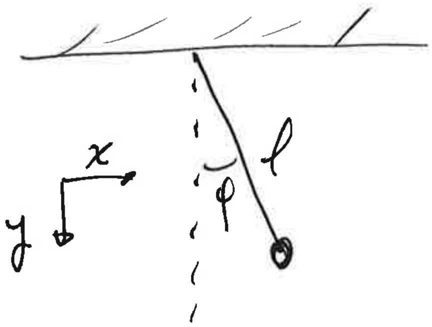
we know $\vec{F}_1 = -\vec{\nabla}_1 U$, $\vec{F}_2 = -\vec{\nabla}_2 U$

in the end, just do it for each particle, 3N coord req.

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1, 2, \dots 3N$$

e.g. COM posn; relative posns

Example: pendulum



- $\ell = \sqrt{x^2+y^2}$ is constant
- φ is only coordinate needed! (only one degree of freedom)

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\dot{\varphi}^2 = \tfrac{1}{2}m\ell^2\dot{\varphi}^2$$

$$U = mgh = mg\ell(1-\cos\varphi)$$

$$\Rightarrow \mathcal{L} = T - U = \tfrac{1}{2}m\ell^2\dot{\varphi}^2 - mg\ell(1-\cos\varphi) = \mathcal{L}(q,\dot{q})$$

- Could start w/ (x,y) but will end in same form
- will prove later: once $\mathcal{L}$ is written in terms of 1 variable for a system w/ 1 degree of freedom, satisfies Euler-Lagrange still

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$\underbrace{-mg\ell\sin\varphi}_{\Gamma \text{ torque due to gravity}} = \underbrace{\frac{d}{dt}(m\ell^2\dot{\varphi})}_{\frac{dL}{dt}} = \underbrace{m\ell^2\ddot{\varphi}}_{I\alpha}$$

just as w/ Newton's laws. Further,

$$\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi \quad \text{as before}$$