constrained systems in general

- arbitrary number of particles $\alpha = 1 \cdots N$ with positions $\vec{r}_\alpha$

then
- parameters $q_1, \cdots, q_n$ are a set of generalized coordinates for the system if each $\vec{r}_\alpha$ can be expressed as a function of $q_1, \cdots, q_n$ and possibly by time t

$$\vec{r}_\alpha = \vec{r}_\alpha(q_1, \cdots, q_n, t) \quad \alpha = 1 \cdots N$$

and conversely each q_i can be expressed in terms of $\vec{r}_\alpha, t$

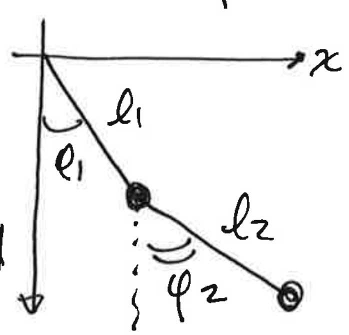
$$q_i = q_i(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N, t) \quad i = 1, \cdots n$$

- want smallest num gen coord n possible to parametrize system

N free particles $\Rightarrow n = 3N$; constraints reduce this

(e.g. rigid body $\sim 10^{23}$ atoms but 6 coord enough - COM posn & body orientation)

e.g. double pendulum. Single pendulum had 2 coord in principle, but only φ needed as coord since length is fixed



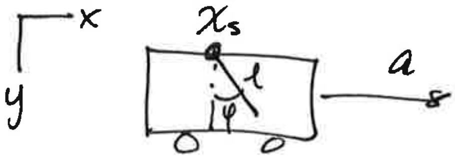
now: 2 bobs in a plane $\Rightarrow$ 4 Cartesian coord

but! $\vec{r}_1 = (l_1 \sin\varphi_1, l_1 \cos\varphi_1) = \vec{r}_1(\varphi_1)$

$$\vec{r}_2 = (l_1 \sin\varphi_1 + l_2 \sin\varphi_2, l_1 \cos\varphi_1 + l_2 \cos\varphi_2) = \vec{r}_2(\varphi_1, \varphi_2)$$

only need 2 generalized coordinates φ_1, φ_2 because l_1, l_2 constrained

Can depend on time. pendulum in accelerating car, $\vec{a}$ given



bob support: $x_s = \frac{1}{2}at^2$

$$\vec{r} \equiv (x,y) = (\underbrace{l\sin\varphi}_{x_{bob}} + \underbrace{\tfrac{1}{2}at^2}_{x_{car}},\ l\cos\varphi) = \vec{r}(\varphi, t)$$

issue here is inertial frame - if we analyze within car, non-inertial frame. We'd need a fictitious force to mimick $\vec{a}$, and Lagrangian won't work. Analyze from POV on ground.

- q_i are natural if they relate to Cartesian $\vec{r}_\alpha$ w/o time dep
- system is holonomic if it has n degrees of freedom and requires n generalized coordinates
  - requiring more? more complicated, we avoid!
  - example - rolling ball w/ mark on top
    need 3 posn + 3 orientation coords
    EVEN though only 3 degrees of freedom

7.4 more general; solid proofs - review on your own

- action is stationary on right path even if there is a constraining force (e.g. normal force to stay on a surface)
- Lagrange eqns work for any holonomic system
  n deg freedom, n coord, non-constraint forces derivable from $U(q_i, t)$

how to find the force? $\mathcal{L} = T - U$. Let $U = U(q)$ indep t, $\dot{q}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} - \frac{\partial U}{\partial q} = 0 \quad \text{but } F_q = -\frac{\partial U}{\partial q}$$

$$\Rightarrow F_q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = m\ddot{q}$$ } for all indep q, as any consv. force is

Examples

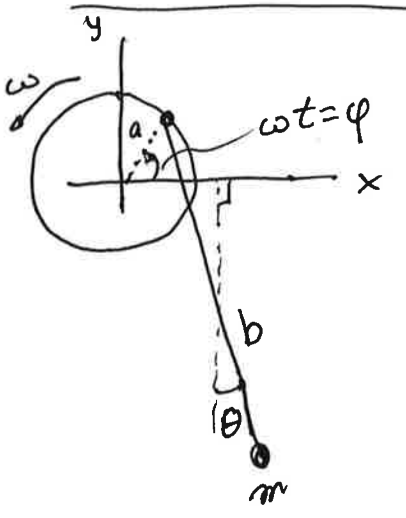
$\boxed{\text{SHO}}$ $T = \frac{1}{2}m\dot{x}^2 \quad U = \frac{1}{2}kx^2 \Rightarrow \mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx = F_x \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} \Rightarrow m\ddot{x} = -kx \ \checkmark$$

$\boxed{\text{gravity}}$ $\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy$

$$\frac{\partial \mathcal{L}}{\partial y} = -mg = F_y \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y} \Rightarrow \ddot{y} = -g \ \checkmark$$

Pendulum on a wheel rotating w/ ω (7.29)



just find coordinates of bob first

$$x = a\cos\omega t + b\sin\theta = x(\theta, t)$$
$$y = a\sin\omega t - b\cos\theta = y(\theta, t)$$
} θ is our gen. coord

need $\dot{x}, \dot{y}$ for T

$$\dot{x} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$$
$$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta \quad \text{chain rule...}$$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = mgy$ so $U(0) = 0$

arbitrary shift would be irrelevant since only derivatives matter

$$\mathcal{L} = T - U = \tfrac{1}{2} m \big(a^2\omega^2 \sin^2\omega t + b^2\dot{\theta}^2\cos^2\theta - 2ab\omega\dot{\theta}\sin\omega t\cos\theta$$
$$+ a^2\omega^2\cos^2\omega t + b^2\dot{\theta}^2\sin^2\theta + 2ab\omega\dot{\theta}\cos\omega t\sin\theta \big) \quad \sin(a-b)$$
$$- mg\,(a\sin\omega t - b\cos\theta)$$

$$\mathcal{L} = \tfrac{1}{2} m \left[a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\dot{\theta}\omega\sin(\theta - \omega t) - mg(a\sin\omega t - b\cos\theta) \right]$$

clearly θ is our coordinate

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mab\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = mb\dot{\theta}a\omega\cos(\theta - \omega t) - mgb\sin(\theta)$$

mb in every term

$$b\ddot{\theta} + \underline{a\omega(\dot{\theta}}-\omega)\cos(\theta - \omega t) = \underline{a\omega\dot{\theta}\cos(\theta - \omega t)} - g\sin\theta$$

$$b\ddot{\theta} = a\omega^2\cos(\theta - \omega t) - g\sin\theta$$

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$$

in principle done - can solve numerically

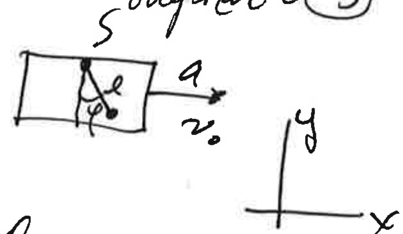
let $\omega \to 0$? $\ddot{\theta} = -\frac{g}{b}\sin\theta$ usual pendulum

simple view: period depends on angle non-simply

imagine doing this with forces! turns out to be chaotic

* Wolfram demo *

origin @ $t=0$



Pendulum in an accel. car from before (7.30)

had $x = \underbrace{v_0 t + \frac{1}{2}at^2}_{\text{car}} + \underbrace{l\sin\varphi}_{\text{bob rel car}}$ $\quad y = -l\cos\varphi$

Choose y up now

$\Rightarrow \dot{x} = v_0 + at + l\dot{\varphi}\cos\varphi \qquad \dot{y} = l\dot{\varphi}\sin\varphi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = -mgl\cos\varphi \qquad$ only φ requ. as gen coord

$\mathcal{L} = \frac{1}{2}m(v_0 + at + l\dot{\varphi}\cos\varphi)^2 + \frac{1}{2}m(l\dot{\varphi}\sin\varphi)^2 + mgl\cos\varphi$

- can drop v_0 - will not survive derivatives anyway
  let $v_0 = 0$, still inertial frame

$\mathcal{L} = \frac{1}{2}m(a^2t^2 + 2atl\dot{\varphi}\cos\varphi + \underline{l^2\dot{\varphi}^2\cos^2\varphi + l^2\dot{\varphi}^2\sin^2\varphi}) + mgl\cos\varphi$

$= \frac{1}{2}m(a^2t^2 + 2atl\dot{\varphi}\cos\varphi + l^2\dot{\varphi}^2) + mgl\cos\varphi$

$\frac{\partial\mathcal{L}}{\partial\varphi} = -matl\dot{\varphi}\sin\varphi - mgl\sin\varphi$

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{d}{dt}(matl\cos\varphi + ml^2\dot{\varphi}) = mal\cos\varphi - matl\dot{\varphi}\sin\varphi + ml^2\ddot{\varphi}$

$\underline{\underline{m}}l^2\ddot{\varphi} + \underline{\underline{m}}al\cos\varphi - \underline{matl\dot{\varphi}\sin\varphi} = -\underline{\underline{m}}gl\sin\varphi - \underline{matl\dot{\varphi}\sin\varphi}$

$l^2\ddot{\varphi} + al\cos\varphi = -gl\sin\varphi$

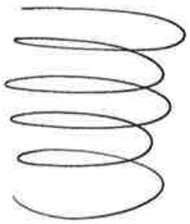
$\ddot{\varphi} = -\frac{a}{l}\cos\varphi - \frac{g}{l}\sin\varphi \qquad$ ($a=0$: regular pendulum)

equ? $\ddot{\varphi} = 0$, or $\underline{g\sin\varphi = -a\cos\varphi} \Rightarrow \tan\varphi_{eq} = -\frac{a}{g}$

physically - as it must be

can also show $\ddot{\varphi} = -\frac{\sqrt{a^2+g^2}}{l}\sin(\varphi + \varphi_{eq})$ ... small osc freq

Particle in a helical wire (7-20)



$\rho = R$
$z = \lambda\varphi$

$v_z = \dot{z} \quad v_\varphi = R\dot{\varphi} \quad \Rightarrow v^2 = \dot{z}^2 + R^2\dot{\varphi}^2$

$T = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\varphi}^2) = \frac{1}{2}m\dot{z}^2\left(1+\frac{R^2}{\lambda^2}\right) = T(\dot{z})$

but $\dot{\varphi} = \frac{1}{\lambda}\dot{z}$ ... constraint of path eliminates two coord (b/c const R gets of one) – 1D problem!

$U = mgz \quad \Rightarrow \mathcal{L} = \frac{1}{2}m\dot{z}^2\left(1+\frac{R^2}{\lambda^2}\right) - mgz = \mathcal{L}(z,\dot{z})$

$\frac{\partial\mathcal{L}}{\partial z} = -mg \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} = \frac{d}{dt}\left(m\dot{z}\left(1+\frac{R^2}{\lambda^2}\right)\right) = m\ddot{z}\left(1+\frac{R^2}{\lambda^2}\right)$

$\Rightarrow \ddot{z} = \dfrac{-g}{1+R^2/\lambda^2}$

$R\to 0$, $a=-g$ – just falling! Otherwise, falls slower than free particle, depends on radius relative to pitch of helix

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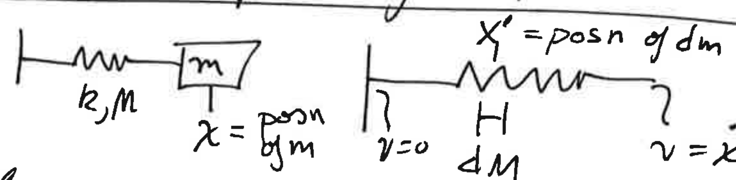
Spring w/ non-negl mass



assume v varies linearly fr spring

$dT_s = \frac{1}{2}v_{dm}^2\,dm \qquad v_{dm} = \left(\frac{x'}{l+x}\right)v$

$dM = \left(\frac{M}{l+x}\right)dx'$
$(l+x)$ = current length
x' = posn along spring

$T_s = \int_0^{l+x} \frac{1}{2}\frac{x'^2v^2}{(l+x)^2}\frac{M}{(l+x)}dx' = \frac{1}{2}\frac{\dot{x}^2M}{(l+x)^3}\int_0^{l+x}x'^2\,dx = \frac{1}{2}\dot{x}^2\frac{M}{(l+x)^3}\left.\frac{x'^3}{3}\right|_0^{l+x}$

over spring

$\Rightarrow T_s = \frac{1}{6}M\dot{x}^2$ as though $\frac{1}{3}$ of M moves w/ $\dot{x} = v_m$

$$\Rightarrow \quad T = T_s + T_m = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{6} M\dot{x}^2 \qquad \mathcal{U} = \tfrac{1}{2} k x^2$$

$$\mathcal{L} = T - \mathcal{U}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx = F_x \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\left(m\dot{x} + \tfrac{1}{3}M\dot{x}\right) = \tfrac{1}{3}M\ddot{x} + m\ddot{x}$$

$$\Rightarrow \; kx + \ddot{x}\left(m + \tfrac{1}{3}M\right) = 0$$

$$\ddot{x} = \left(\frac{k}{m + M/3}\right)x \qquad \Rightarrow \text{SHM w/ } \omega = \sqrt{\frac{k}{m + M/3}}$$

- effective mass includes $\frac{1}{3}$ of spring mass, lowers freq.
- is NOT damping though as you might have expected