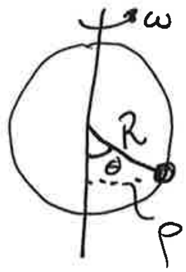
PH301
L19 F18

example 7.6 - Bead on a spinning wire hoop

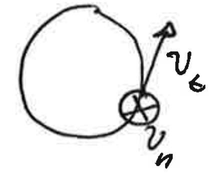


$\rho = R\sin\theta$
$\omega = \dot{\varphi}$

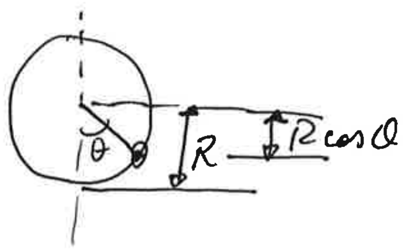
- with R, $\dot{\varphi}$ fixed, only need θ as generalized coordinate
- 3dim - 2 fixed coord = 1D!

velocity? tangential to hoop, $v_t = R\dot{\theta}$

normal to hoop $v_n = \rho\omega = (R\sin\theta)\omega$



$\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_t^2 + v_n^2) = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta)$



$h = R - R\cos\theta \Rightarrow U = mgR(1-\cos\theta)$

$\mathcal{L} = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR(1-\cos\theta)$

$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \Rightarrow mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta = \frac{d}{dt}(mR^2\dot{\theta})$
$= mR^2\ddot{\theta}$

cancel mR^2: $\ddot{\theta} = \sin\theta(\omega^2\cos\theta - g/R)$

- not easily solved, but can still find out behavior
- at an eqn point θ_0, if bead is placed at rest ($\dot{\theta}=0$) it will remain at rest - if $\dot{\theta}=0$, $\theta=\theta_0$, then $\ddot{\theta}=0$ so $\dot{\theta}$ and θ remain constant
- for $\ddot{\theta}=0$

$$(\omega^2\cos\theta - \tfrac{g}{R})\sin\theta = 0$$

true if either is zero

- $\sin\theta = 0$ when $\theta = 0$ or π ⇒ bead can remain @ rest at top or bottom of hoop

- when $\cos\theta = \frac{g}{\omega^2 R}$ in equ.; since $|\cos\theta| \leq 1$,
  ⇒ can only vanish when $\omega^2 \geq g/R$
  ⇒ 2 equ @ $\theta_0 = \pm \arccos\left(\frac{g}{\omega^2 R}\right)$

- when hoop rotates slowly ($\omega^2 \leq g/R$) only top & bottom are equ posns

- fast enough? $\omega^2 \geq g/R$ 2 new equ posns, symmetric on either side of bottom

are they stable?

near $\theta = 0$, $\cos\theta \simeq 1$ & $\sin\theta \simeq \theta$

⇒ $\ddot{\theta} \simeq (\omega^2 - g/R)\,\theta$

- if $\omega^2 < g/R$ ⇒ $\ddot{\theta} = -(\text{const})\,\theta$
  ⇒ nudge to RHS, accelerates back to LHS
  restoring force: SHM @ freq $\left(\frac{g}{R} - \omega^2\right)$
- faster: $\ddot{\theta} = +(\text{const})\,\theta$
  nudge to RHS, accelerates to RHS
  bottom equ unstable, heads to $+\theta_0$

- at top $\theta = \pi$ always unstable

how about side equ. points when $\omega^2 > g/R$?

$$\ddot{\theta} = (\omega^2 \cos\theta - g/R)\sin\theta$$

expect equ $0 \le \theta_0 \le \frac{\pi}{2}$

- at θ_0, ( ) = 0, $\sin\theta$ is +
- incr θ? $\sin\theta \uparrow$, $\cos\theta \downarrow \Rightarrow \ddot{\theta} < 0$, restoring! Stable!

- slow rot, stable at $\theta = 0$. Above crit speed $\omega^2 = g/R$, bottom is unstable, bead shifts to $\pm\theta_0$
  - used as a governor on early steam engines - too fast, bead shifts up and shuts off steam
- Show Wolfram demonstration

Generalized Momenta & Ignorable Coordinates

$$\underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\text{gen force}} = \frac{d}{dt} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\text{gen mom.}} \quad \Leftrightarrow \quad F_i = \frac{d}{dt} p_i \Rightarrow$$

if $\mathcal{L}$ is indep of coordinate q_i, then $F_i = \frac{\partial \mathcal{L}}{\partial q_i} = 0$, so momentum p_i is const.

e.g. projectile $\mathcal{L} = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

gen forces $\frac{\partial \mathcal{L}}{\partial x} = F_x = -\frac{\partial U}{\partial x}$ etc

gen mom $\frac{\partial \mathcal{L}}{\partial \dot{x}} = p_x$

but $\mathcal{L}$ is indep x, y, so p_x & p_y are constant (as we know)

2D polar: gives torque & angular momentum

- When a particle's Lagrangian is indep of coordinate q_i we say that coordinate is <u>ignorable</u> or cyclic
- key task when choosing gen coord: find as many as possible that are ignorable!
- rephrase: if $\mathcal{L}$ is invariant under variations of q_i, the corresponding generalized momentum p_i is conserved - invariance $\Leftrightarrow$ conservation law
  Noether's theorem

7.8 Conservation of momentum

<u>translationally invariant system</u>: move all particles through the same displacement $\vec{\epsilon}$, nothing physically significant changes

i.e. $\vec{r}_\alpha \to \vec{r}_\alpha + \vec{\epsilon}$, $\vec{r}_1 \to \vec{r}_1 + \vec{\epsilon}$, ... $\vec{r}_N \to \vec{r}_N + \vec{\epsilon}$

PE must be unaffected since it depends only on <u>relative</u> spacing of particles, which doesn't change

$$U(\vec{r}_1 + \vec{\epsilon}, \cdots, \vec{r}_N + \vec{\epsilon}, t) = U(\vec{r}_1, \cdots, \vec{r}_N, t)$$

$$\text{or} \quad \delta U = 0 = U_{trans} - U_{orig}$$

adding const $\vec{\epsilon}$ doesn't change T either

$$\dot{\vec{r}}_\alpha \to \dot{\vec{r}}_\alpha + \dot{\vec{\epsilon}} = \dot{\vec{r}}_\alpha \qquad \Rightarrow \delta \mathcal{L} = 0$$

but: $\delta \mathcal{L} = \epsilon \frac{\partial \mathcal{L}}{\partial x_1} + \epsilon \frac{\partial \mathcal{L}}{\partial x_2} + \cdots \epsilon \frac{\partial \mathcal{L}}{\partial x_n}$ $\left(\delta f = \delta x \frac{\partial f}{\partial x} \right)$

(can choose $\vec{\epsilon}$ along x w/o loss of generality,)
or first consider x, y, z separately

or $\sum_{\alpha=1}^{N} \frac{\partial \mathcal{L}}{\partial x_\alpha} = 0$ now use Lagrange's eqns

$$\frac{\partial \mathcal{L}}{\partial x_\alpha} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha} = \frac{d}{dt} p_{\alpha x}$$ x component of $\vec{p}$ for particle α

$$\Rightarrow \sum_{\alpha=1}^{N} \frac{d}{dt} p_{\alpha x} = \frac{d}{dt} P_x = 0$$

where $P_x = \sum_\alpha p_x$, $\vec{P} = \sum_\alpha \vec{p}_\alpha$

x comp of total momentum — total momentum

Can do the same for components of $\vec{\epsilon}$ along y, z
$\Rightarrow$ total momentum is conserved if the system is translationally invariant

symmetry	conserved quantity
time invariance	total energy
rotational invariance	angular momentum
translational invariance	linear momentum

quantum field theory - cons. electric charge

Conservation of energy

- As time advances, $\mathcal{L}(q_1, \dots q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ changes both because t changes and because q_i and $\dot{q}_i$ do
- Chain rule:

$$\frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i, t) = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

Using Lagrange eqn, $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt}p_i = \dot{p}_i$

Second sum is just generalized momentum p_i

$$\Rightarrow \quad \frac{d}{dt}\mathcal{L} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\left(\sum_i p_i \dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t}$$

In many systems, $\mathcal{L}$ is indep of t, so $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$\Rightarrow \quad \frac{d}{dt}\mathcal{L} = \frac{d}{dt}\sum_i p_i \dot{q}_i \quad \text{or} \quad \frac{d}{dt}\left(\sum_i p_i \dot{q}_i - \mathcal{L}\right) = 0$$

$$\Rightarrow \quad \sum_i p_i \dot{q}_i - \mathcal{L} = \text{const} \equiv \mathcal{H}$$ Hamiltonian of the sys

- if $\mathcal{L}$ does not depend explicitly on time, $\frac{\partial \mathcal{L}}{\partial t} = 0$, the Hamiltonian $\mathcal{H}$ is conserved
- if the relationship btw generalized and Cartesian coordinates is time-indep, $\vec{r}_\alpha = \vec{r}_\alpha(q_i)$, the Hamiltonian is just total energy

$$\mathcal{H} = T + U$$

- another view: $\mathcal{L}$ unchanged by translations in time $t \rightarrow t + \epsilon$