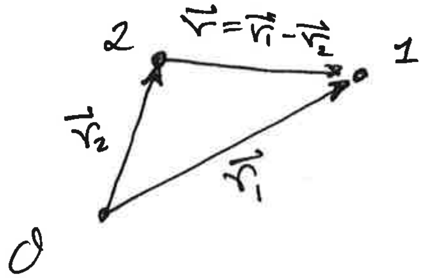
(ph301 F18
L20 ch8

- 2 bodies, each of which exerts a central, conservative force on the other

- usually an approx (e.g. ignore other planets; worry about earth-sun)

The problem: m_1 and m_2 at $\vec{r}_1$ and $\vec{r}_2$.



- only forces are due to their interaction, $\vec{F}_{12}$ and $\vec{F}_{21}$

- assume conservative and central
  $\Rightarrow$ both derivable from $U(\vec{r}_1, \vec{r}_2)$

e.g. gravitation $U(\vec{r}_1, \vec{r}_2) = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$ (earth-sun, earth-moon)

electrical $U(\vec{r}_1, \vec{r}_2) = -\dfrac{ke^2}{|\vec{r}_1 - \vec{r}_2|}$ $\left(k = \frac{1}{4\pi\epsilon_0}\right)$ $(p^+ - e^-)$

- in both cases U depends only on $\vec{r}_1 - \vec{r}_2$
  - any isolated system is translationally invariant
  - if translationally invariant, can only dep. on $\vec{r}_1 - \vec{r}_2$

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

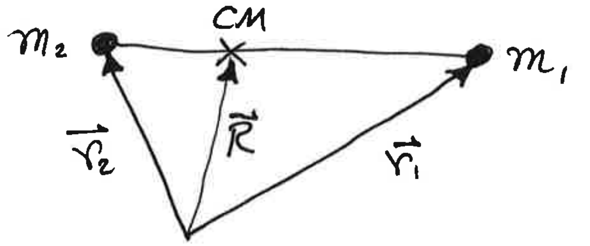
- this means only $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ is relevant, so define $\vec{r}$

$$U = U(\vec{r})$$

Problem: possible motion of 2 bodies whose Lagrangian is

$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$ (can do in Newtonian formalism)

CM and relative coordinates, reduced mass



- Which generalized coord to use?
- clearly $\vec{r}$ should be one as it simplifies U greatly
- but: 2D problem, need one more

- Best choice turns out to be CM

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

let $M = m_1 + m_2$

- As shown above, $m_2 >> m_1$ since COM is close to m_2 (~1:3 ratio)
- We already know total momentum from R, M

$$\vec{P} = M\dot{\vec{R}}$$

but total momentum must be constant for 2 isolated bodies

$\Rightarrow$ $\dot{\vec{R}}$ = const, so CM frame is <u>inertial</u> (no accel)

further: possible to chose an inertial frame where CM is at rest (drive alongside it...)

- this inertial frame w/ CM @ rest is the <u>CM frame</u> and it is very convenient

- So $\vec{r}, \vec{R}$ are our gen coord, $U = U(r)$ still
  to get T, need to relate $\vec{r}_1, \vec{r}_2$ to $\vec{r}, \vec{R}$

note $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$ $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$ $\quad$ combine, w/ $\vec{r}_2 = \vec{r}_1 - \vec{r}$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_1 - m_2\vec{r}}{M} = \frac{(m_1+m_2)\vec{r}_1 - m_2\vec{r}}{M} = \vec{r}_1 - \frac{m_2}{M}\vec{r}$$

$$\Rightarrow \boxed{\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}}$$

now $\dot{\vec{r}}_1 = \dot{\vec{R}} + \dfrac{m_2}{M}\dot{\vec{r}}, \quad \dot{\vec{r}}_2 = \dot{\vec{R}} - \dfrac{m_1}{M}\dot{\vec{r}}$

$$T = \frac{1}{2}\left(m_1\dot{\vec{r}}_1^{\,2} + m_2\dot{\vec{r}}_2^{\,2}\right) = \frac{1}{2}\left[m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)\right]$$

$$= \frac{1}{2}\left[m_1\left(\dot{R}^2 + \underline{\frac{2m_2}{M}\dot{r}\dot{R}} + \frac{m_2^2}{M^2}\dot{r}^2\right) + m_2\left(\dot{R}^2 - \underline{\frac{2m_1}{M}\dot{r}\dot{R}} + \frac{m_1^2}{M^2}\dot{r}^2\right)\right]$$

$$= \frac{1}{2}\left[\dot{R}^2(m_1+m_2) + \dot{r}\dot{R}\left(\frac{2m_1m_2}{M} - \frac{2m_1m_2}{M}\right) + \dot{r}^2\underbrace{\frac{m_1m_2^2 + m_1^2m_2}{M^2}}_{\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}}\right]$$

$$\Rightarrow T = \frac{1}{2}\Big(\underbrace{M\dot{R}^2}_{\text{COM } v} + \underbrace{\frac{m_1m_2}{M}\dot{r}^2}_{\text{rel } v}\Big)$$

now let $\mu = \dfrac{m_1m_2}{M} = \dfrac{m_1m_2}{m_1+m_2}$

"reduced mass"

- reduced mass always $< (m_1 \text{ or } m_2)$; $m_1 \ll m_2$, $\mu \approx m_1$

e.g. earth-sun: $\mu \sim m_e$, sun $\approx$ fixed revising $T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$

same as 2 ficticious particles:

- 1 moving at speed of CM, mass M
- 1 w/mass μ moving w/ speed of relative pos $\vec{r}$

Look @ Lagrangian, even better

$$\mathcal{L} = T - U = \underbrace{\tfrac{1}{2}M\dot{R}^2}_{CM} + \underbrace{\tfrac{1}{2}\mu\dot{r}^2 - U(r)}_{rel} = \mathcal{L}_{cm} + \mathcal{L}_{rel}$$

We've used generalized coords to split $\mathcal{L}$ into 2 pieces, each of which only involves a single coord

$\Rightarrow$ can solve for motions of $\vec{R}$ and $\vec{r}$ separately

huge simplification

8.3 equations of motion

$\mathcal{L}$ is independent of R, so the R eqn is simple

$$\frac{\partial \mathcal{L}}{\partial R} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{d}{dt}(M\dot{R}) = M\ddot{R} \Rightarrow \dot{R} = \text{const}$$

- direct consequence of conservation of total P we already derived
- also reflects $\mathcal{L}$ indep R - makes R ignorable, so P_R cons.
- Lastly, $\mathcal{L}_{cm}$ is the same as a free particle, so from Newton's 1st law we require $\dot{R} = \text{const}$
- CM moves @ const vel, can chose frame where it is 0

Lagrange eqn for $\vec{r}$ is almost as simple, but more astonishing: same math as a single particle of mass μ, posn $\vec{r}$, w/ potential energy $U(r)$. In 3D:

$$\frac{\partial \mathcal{L}}{\partial r} = -\vec{\nabla} U(\vec{r}) \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}(\mu \dot{r}) = \mu \ddot{r}$$

$$\Rightarrow \mu \ddot{r} = -\nabla U(r)$$

So for relative motion, solve Newton's 2nd law for a single particle w/ reduced mass μ and posn $\vec{r}$, w/ PE $U(r)$

CM ref frame

- Since $\dot{R}$ = const, we can choose a frame where CM is at rest $\equiv$ CM frame. As though we're running alongside CM at constant velocity
- In this frame $\dot{R} = 0$, so the CM Lagrangian is zero
- We don't really care what CM does if we know it has constant velocity

- In CM frame, both particles move w/ equal and opposite momenta

- if $m_2 \gg m_1$, CM is close to m_2 so v_2 is small
  in this case, posn $m_2 \simeq$ fixed (Sun-earth)
  can think of a single body m_1 orbiting a fixed center

- In all cases, we get posn of 1 relative to 2, can map back to

<u>Consv L</u> conserved, we know this

$$\vec{L} = (\vec{r}_1 \times \vec{p}_1) + (\vec{r}_2 \times \vec{p}_2) = (m_1 \vec{r}_1 \times \dot{\vec{r}}_1) + (m_2 \vec{r}_2 \times \dot{\vec{r}}_2)$$

in CM frame w/ $\vec{R}=0$, $\vec{r}_1 = \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = -\frac{m_1}{M}\vec{r}$

Substitute ...

$$\Rightarrow L = \left(m_1 \frac{m_2}{M}\vec{r} \times \frac{m_2}{M}\dot{\vec{r}}\right) + \left(m_2\left(\frac{-m_1}{M}\vec{r}\right) \times \left(\frac{-m_1}{M}\dot{\vec{r}}\right)\right)$$

$$= \frac{m_1 m_2}{M^2}\left(m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}}\right)$$

$$= \frac{\mu}{M}\underbrace{(m_1+m_2)}_{M}(\vec{r} \times \dot{\vec{r}}) = \underline{\underline{\vec{r} \times \mu\dot{\vec{r}} = \vec{L}}}$$

- L in CM frame is that of a single particle, mass μ posn $\vec{r}$!
- 2 body problem $\Rightarrow$ $\left\{\begin{array}{l} 1 \text{ trivial } 1 \text{ body (cm)} \\ + 1 \text{ trivial } 1 \text{ body (rel)} \end{array}\right.$

implies by L consv that $\vec{r} \times \dot{\vec{r}} = $ const

... meaning the direction is const for $\vec{r} \times \dot{\vec{r}}$

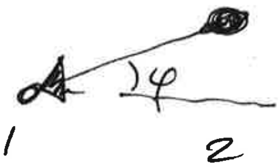
... meaning $\vec{r}, \dot{\vec{r}}$ lie in a fixed plane (say, xy)

$\Rightarrow$ this is a 2D problem, and it maps to a 1 particle problem.

−1 dimension, −1 particle !!

Equations of motion

- remaining 2D problem: use (r, φ) coord



  – $\mathcal{L}$ is independent of φ, so φ is ignorable

$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r) = \frac{1}{2}\mu(\dot r^2 + r^2\dot\varphi^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial \dot\varphi} = \text{const} = \mu r^2 \dot\varphi = \ell \qquad \left(\text{because } \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} = 0\right)$$

so φ eqn just gives back consv $\vec{\ell}$

Radial eqn

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} \Rightarrow \mu r \dot\varphi^2 - \frac{dU}{dr} = \mu \ddot r$$

from ch 7:

$$-\frac{dU}{dr} = \mu\ddot r - \mu r\dot\varphi^2 = \mu \ddot r - \frac{\ell^2}{\mu r^3}$$

$$\vec{F} = m\vec{a} + \vec{F}_{\text{centr}}\left(mr\omega^2 \text{ or } \frac{mv^2}{r}\right) \quad \text{centrifugal}$$

effectively 1D problem w/ centripetal force. As though particle moves in 1D w/ $F = -\nabla U$ <u>plus</u> a fictitious centrifugal force

Next: state precisely the 1D equivalent

- conservation of energy
- allowed orbits
- <u>bound</u> orbits