Central force motion

Ph301 F18
L21 8.3-4
part of 8.5

Equations of motion

$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$ indep $\varphi \Rightarrow \varphi$ is ignorable

$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \text{const} = \mu r^2 \dot{\varphi} = \ell$ cons angular momentum

$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \Rightarrow \mu r \dot{\varphi}^2 - \frac{\partial U}{\partial r} = \mu \ddot{r}$

noting $\dot{\varphi} = \frac{\ell}{\mu r^2}$, we can eliminate $\dot{\varphi}$ above

$$\mu \ddot{r} = -\frac{dU}{dr} + \mu r \dot{\varphi}^2 = -\frac{dU}{dr} + F_{cf}$$

- looks like $F = ma$ in 1D, but total force is actual force $-\frac{dU}{dr}$ + "fictitious" outward centrifugal force $F_{cf} = \mu r \dot{\varphi}^2 = \mu v_\varphi^2 / r$
- eliminating $\dot{\varphi}$, $F_{cf} = \mu r \left(\frac{\ell}{\mu r^2}\right)^2 = \frac{\ell^2}{\mu r^3}$
- Can express F_{cf} in terms of a centrifugal potential

$$F_{cf} = -\frac{dU_{cf}}{dr} = \frac{\ell}{\mu r^3} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right)$$

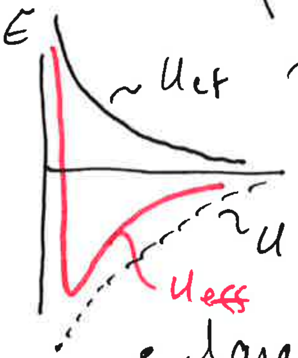
so $U_{cf}(r) = \frac{\ell^2}{2\mu r^2}$

- radial eqn is now

$$\mu \ddot{r} = -\frac{d}{dr}\left[U(r) + U_{cf}(r)\right] = -\frac{d}{dr} U_{eff}(r)$$

radial motion is equivalent to a 1D particle moving in an *effective potential* $U_{eff} = U + U_{cf}$ — required to make Newton's laws work in accel frame of particle

Example: comet orbiting Sun



$$U(r) = -\frac{Gm_1 m_2}{r} \Rightarrow U_{eff}(r) = \underbrace{-\frac{Gm_1 m_2}{r}}_{\text{attractive}} + \underbrace{\frac{\ell^2}{2\mu r^2}}_{\text{repulsive}}$$

- large r, $\frac{\ell^2}{2\mu r^2}$ is negligible compared to gravitational term
  - $\Rightarrow$ U_{eff} is negative and sloping up
  - $\Rightarrow$ $\ddot{r}$ is always inward toward Sun – attractive
- Small r, centrifugal term dominates
  - $\Rightarrow$ near $r=0$, U_{eff} is positive and slopes downward – repulsive
  - $\Rightarrow$ as comet approaches sun, $\ddot{r}$ eventually becomes outward and comet moves away from Sun
- exception: if $\ell = 0$, no centrifugal force, $\dot{\varphi} = 0$
  Comet is moving radially along a line of constant φ, and will hit the sun

Conservation of energy

look at radial eqn $\mu \ddot{r} = -\frac{d}{dt} U_{eff}(r)$

multiply both sides by $\dot{r}$, note

$$\frac{d}{dt}\left(\tfrac{1}{2}\mu \dot{r}^2\right) = \mu \ddot{r} = \frac{d}{dt}\left(U_{eff}(r)\right)$$

or $\quad \tfrac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = \text{const} \qquad$ cons E

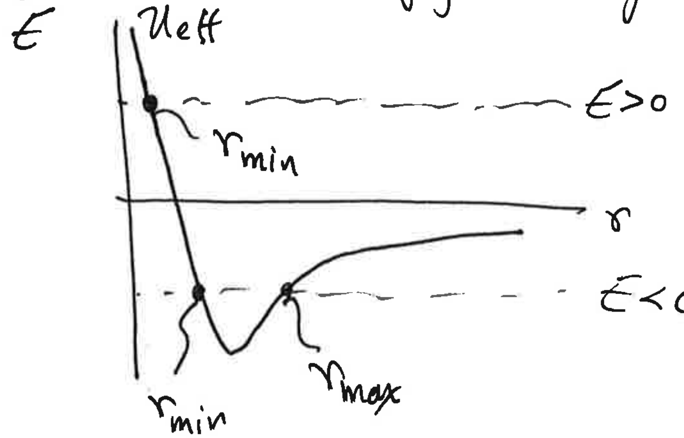
specifically, $\quad \tfrac{1}{2}\mu \dot{r}^2 + \tfrac{1}{2}\mu r^2 \dot{\varphi}^2 + U(r) = E = \text{const}$

$\left(\ell^2/2\mu r^2 = \tfrac{1}{2}\mu r^2 \dot{\varphi}^2\right.$ recall$\left.\right)$

- 2D problem is equivalent to 1D problem involving just the radial motion
- effective potential includes actual potential $U(r)$ and kinetic E of angular motion $T_\varphi = \tfrac{1}{2}\mu r^2 \dot{\varphi}^2$
- can now solve like any other 1D problem

Energy for a planet or comet

- given total energy E, confined to regions where $E \geq U_{eff}$

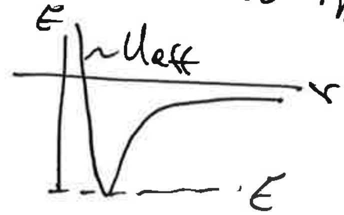


- first case, $E > 0$
- cannot get closer than r_{min} where $U_{eff}(r_{min}) = E$
- moving toward sun? get to r_{min} where $\dot{r} = 0$ then moves outward

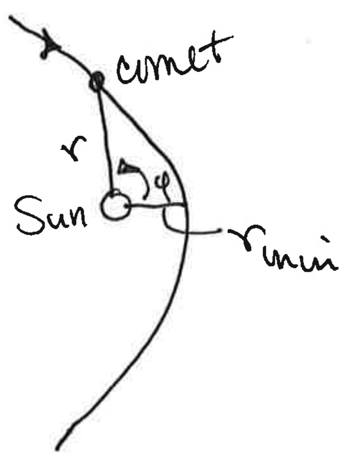
- for $E>0$, no other points where $\dot{r}$ can vanish
  has enough energy to go to $r \to \infty$ $\Rightarrow$ unbounded
- for $E<0$, two turning points – no closer than r_{min}
  – no farther than r_{max}

  $\Rightarrow$ bounded orbit, oscillates btw r_{min} and r_{max}

  - if E = min of U_{eff}, only one value of r is allowed! $\Rightarrow$ circular orbit



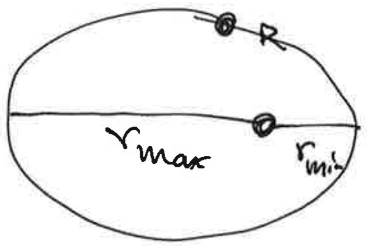
- not just gravity – similar for a diatomic molecule, e.g.
- unbounded comet? $E>0$ note $\dot{\varphi} = \ell/\mu r^2$

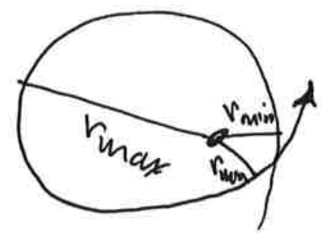


  - approaching, φ changes at a rate that increases as $r\downarrow$
  - leaving: φ changes at a rate that decreases as $r\uparrow$

- bounded? r osc. between r_{min} & r_{max}, period is the time it takes for φ to make 1 complete revolution if the law is $F \sim 1/r^2$

  – other forces, no guarantee orbit is closed





$F \sim 1/r^2$

most other force laws: orbit doesn't close after 1 cycle

Equation of the orbit

$$\mu \ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

F = actual force $= -\frac{dU}{dr}$

$\frac{\ell^2}{\mu r^3}$ = centrifugal $= -\frac{dU_{cf}}{dr}$

- Can solve w/ some clever substitutions
- rewrite in terms of φ, use sub.

$$u = \frac{1}{r} \quad \text{or} \quad r = \frac{1}{u}$$

- $$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi} = \frac{\ell}{\mu r^2}\frac{d}{d\varphi} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}$$

chg var $u = \frac{1}{r}$

now $\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\varphi}$ $\quad \left(\frac{d}{d\varphi}\frac{1}{u} = \frac{-1}{u^2}\frac{du}{d\varphi}\right)$

so $\ddot{r} = \frac{d}{dt}(\dot{r}) = \underbrace{\frac{\ell u^2}{\mu}\frac{d}{d\varphi}}_{d/dt}\left(-\frac{\ell}{\mu}\frac{du}{d\varphi}\right) = \frac{-\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2}$

back to radial eqn $\Rightarrow$ $$-\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2} = F + \frac{\ell^2 u^3}{\mu}$$

or $u''(\varphi) = -u(\varphi) - \left(\frac{\mu}{l^2 u(\varphi)^2}\right) F$ solve for a given F to get $u(\varphi) \to \frac{1}{r(\varphi)}$

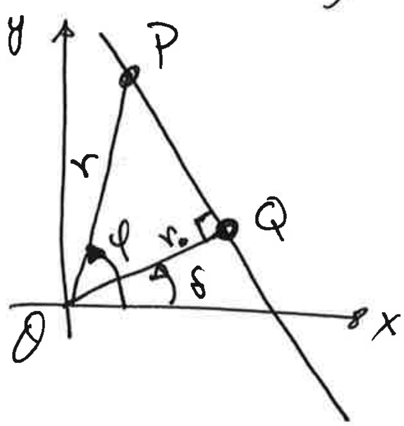
- will give eqn of orbits!
- if $F \propto \frac{1}{r^2}$, conic sections!

Free particle $F = 0$, $\Rightarrow$ $u''(\varphi) = -u(\varphi)$

- know this one! $u(\varphi) = A\cos(\varphi - \delta)$ $A, \delta =$ arb. const

let $A = \frac{1}{r_0}$ $\Rightarrow$ $r(\varphi) = \frac{1}{u(\varphi)} = \frac{r_0}{\cos(\varphi - \delta)}$

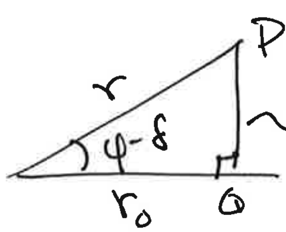
this is actually a straight line in polar coordinates!



let Q = fixed point $= (r_0, \delta)$

line is through Q and $\perp$ to OQ

$P = (r, \varphi)$ lies on the line if and only if $r\cos(\varphi - \delta) = r_0$



next step: figure it out for an inverse square law