- had an equation of motion for $\ddot{r}$
- recast in terms of new variable $u = \frac{1}{r}$
- changed time derivatives to φ derivatives
- goal: $u(\varphi) = \frac{1}{r(\varphi)}$ polar eqn of orbit

$$\frac{d^2u}{d\varphi^2} = -u(\varphi) - \frac{\mu}{\ell^2[u(\varphi)]^2} F \quad \text{where } F(u) \text{ is the force law}$$

- clearly if $F \propto u^2 = \frac{1}{r^2}$ the eqn is simple,
  form $u'' = -u + C$ choice of origin eliminates C
  $\Rightarrow$ SHM eqn!

So let $F = \frac{-\gamma}{r^2} = -\gamma u^2 \Rightarrow \frac{d^2u}{d\varphi^2} = -u(\varphi) - \underbrace{\frac{\gamma\mu}{\ell^2}}_{\text{const}}$

now shift origin: let $W(\varphi) = u(\varphi) - \frac{\gamma\mu}{\ell^2}$, $w'' = u'' \Rightarrow \underline{\underline{w'' = -w}}$

$\Rightarrow w(\varphi) = A\cos(\varphi - \delta)$ known soln!

$\Rightarrow u(\varphi) = A\cos(\varphi - \delta) + \frac{\gamma\mu}{\ell^2} = \frac{\gamma\mu}{\ell^2}\left(1 + \frac{A\ell^2}{\gamma\mu}\cos(\varphi - \delta)\right)$

- define $\epsilon \equiv \frac{A\ell^2}{\gamma\mu}$, $c = \ell^2/\gamma\mu$
- Choose φ origin such that $\delta = 0$

$\Rightarrow u(\varphi) = \frac{1}{c}(1 + \epsilon\cos\varphi) \Rightarrow \boxed{r(\varphi) = \frac{c}{1 + \epsilon\cos\varphi}}$

note $c = r(\varphi = 90°)$

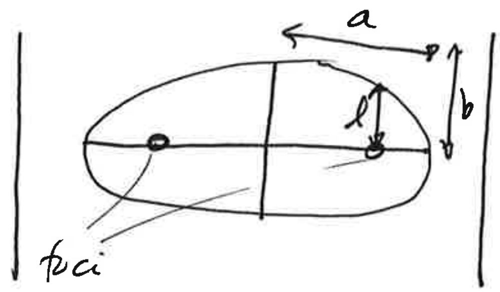
for gravity, $\gamma = Gm_1m_2 = G\mu M$

$\Rightarrow c = \frac{\ell^2}{G\mu^2 M} = [\text{distance}] \quad \epsilon = \frac{A\ell^2}{G\mu^2 M} = [\text{dimensionless}]$

what are these curves? recall $r = \frac{\ell}{1+\epsilon\cos\varphi}$ = polar expr for conic sections!

ϵ = eccentricity = $\begin{cases} 0 & \text{circle} \\ 0<\epsilon<1 & \text{ellipse} \\ \epsilon=1 & \text{parabola} \\ \epsilon>1 & \text{hyperbola} \end{cases}$ circle, ellipse: bound, closed; parabola, hyperbola: unbound

ℓ = semilatus rectum



directrices

$\epsilon < 1$

a = semimajor axis
b = semiminor axis
$\epsilon = \sqrt{1 - b^2/a^2}$
ℓ = semilatus rectum $= \frac{b^2}{a}$
= $\frac{1}{2}$ of chord through focus parallel to directrix
$= \ell^2 / G\mu^2 M$

Bounded orbits

forget we identified ϵ and $c = \ell$
what are the properties of the orbits for $0 \le \epsilon < 1$?

$$r = \frac{c}{1+\epsilon\cos\varphi}$$

- for $\epsilon<1$, the denominator never vanishes
  - so $r(\varphi)$ is bounded for all φ
  - due to $\cos\varphi$ term, $r(0) = r(2\pi)$ orbits are closed in 1 revolution
- if $\epsilon \ge 1$ denominator vanishes for some φ so $r \to \infty$ as φ approaches that angle - unbounded!
- $\epsilon = 1$ is the boundary btw closed & unclosed orbits

with $\epsilon<1$, $\cos\varphi$ has min/max of ∓ 1, so $r(\varphi)$ oscillates between $r_{min} = \frac{c}{1+\epsilon}$ $\quad r_{max} = \frac{c}{1-\epsilon}$
perihelion, $\varphi = 0$ $\quad$ aphelion, $\varphi = \pi$

What is the shape, if we didn't recognize polar form of conic sections?

(long detour)

note $r\cos\varphi = x$, and $r(1+\epsilon\cos\varphi) = c$

$$\Rightarrow r(1+\epsilon\cos\varphi) = r(1+\epsilon x) = c \quad \text{or} \quad r = c - \epsilon x$$

$$r^2 = x^2+y^2 = (c-\epsilon x)^2 = c^2 - 2c\epsilon x + \epsilon^2 x$$

$$\Rightarrow c^2 = (1-\epsilon^2)x^2 + 2c\epsilon x + y^2 \qquad \text{divide by } 1-\epsilon^2$$

$$\frac{c^2}{1-\epsilon^2} = x^2 + \frac{c\epsilon}{1-\epsilon^2}(2x) + \frac{y^2}{1-\epsilon^2} \qquad \text{let } \boxed{d = \frac{c\epsilon}{1-\epsilon^2}}$$

$$\frac{c^2}{1-\epsilon^2} = x^2 + 2dx + \frac{y^2}{1-\epsilon^2}$$

- looks like $x^2+2dx+d^2\ldots$
- complete square by adding d^2 to both sides

$$x^2+2dx+d^2+\frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + \underbrace{\frac{c^2\epsilon^2}{1-\epsilon^2}}_{d^2}$$

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}\left(1+\frac{\epsilon^2}{1-\epsilon^2}\right) = \frac{c^2}{1-\epsilon^2}\left(\frac{1-\epsilon^2+\epsilon^2}{1-\epsilon^2}\right) = \frac{c^2}{(1-\epsilon^2)^2}$$

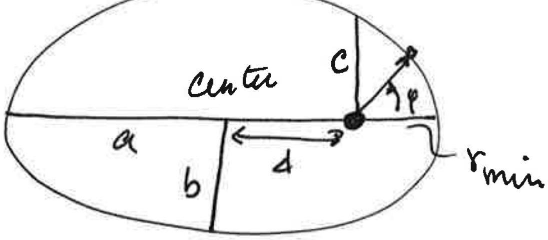
now let $\boxed{a = \frac{c}{1-\epsilon^2}}$ and note $\boxed{d = a\epsilon}$

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = a^2 \Rightarrow \frac{(x+d)^2}{a^2} + \frac{y^2}{a^2(1-\epsilon^2)} = 1$$

now let $\boxed{b = a\sqrt{1-\epsilon^2}}$

$$\Rightarrow \frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipse in cartesian coord! bound orbits for $\epsilon < 1$ are ellipses; $\epsilon = 0$ special case is a circle



orbit is an ellipse with one body at a focus

$a+d = r_{max}$

c = semilatus rectum $= r(\varphi=90°) = \dfrac{\ell^2}{\mu\gamma}$

$a = \dfrac{c}{1-\epsilon^2} \qquad b = \dfrac{c}{\sqrt{1-\epsilon^2}} \qquad d = a\epsilon$

$\dfrac{b}{a} = \sqrt{1-\epsilon^2}$ ← one definition of eccentricity

$\dfrac{(x+d)^2}{a^2} + \dfrac{y^2}{b^2} = 1$

a = semimajor axis
b = semiminor axis

<u>Orbital period</u> recall $\dfrac{dA}{dt} = \dfrac{\ell}{2\mu}$ = constant

and for an ellipse $A = \pi ab$

period is then $\tau = \dfrac{A}{dA/dt}$ (like $t = \frac{x}{\dot{x}}$ when $\ddot{x}=0$)

$\tau = \dfrac{\pi ab}{\ell/2\mu} = \dfrac{2\pi ab\mu}{\ell}$ square, note $b^2 = a^2(1-\epsilon^2)$

$c = a(1-\epsilon^2)$

$\tau^2 = \dfrac{4\pi^2 a^2 b^2 \mu^2}{\ell^2} = \dfrac{4\pi^2 a^4(1-\epsilon^2)\mu^2}{\ell^2} = 4\pi^2 \dfrac{a^3 c\mu^2}{\ell^2}$ but $c = \ell^2/\gamma\mu$

$\Rightarrow \tau^2 = 4\pi^2 \dfrac{a^3\mu}{\gamma}$ for gravity, $\gamma = Gm_1m_2 = G\mu M$

$\Rightarrow \boxed{\tau^2 = 4\pi^2 \dfrac{a^3}{GM}}$ but for a system like the earth & sun, $M \simeq M_S$ earth mass negligible

$\Rightarrow \boxed{\tau^2 = \dfrac{4\pi^2 a^3}{GM_S}}$

- Same for <u>any</u> body orbiting the sun
- Kepler's 3rd law $\tau^2 \propto a^3$
- known τ, a, indep measurement of G allows determination of solar mass!

Same for sattelites of earth. for low earth orbit, height above surface $h << $ radius of earth R_e

low earth orbit, $a \simeq R_e = 6.38 \times 10^6 \, m$

$$\tau = 2\pi \sqrt{\frac{R_e^3}{GM_e}} = 2\pi \sqrt{\frac{R_e}{g}} \simeq 85 \text{ min}$$

low earth orbits are all $\sim 1\frac{1}{2}$ hours

Eccentricity & Energy

- at closest approach, r_{min}, $E = U(r_{min})$ and $T = 0$

$$E = U_{eff}(r_{min}) = \frac{-\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

note $r_{min} = \frac{c}{1+\epsilon} = \frac{\ell^2}{\mu\gamma(1+\epsilon)}$

$$\Rightarrow E = \frac{\mu\gamma(1+\epsilon)}{2\ell^2}\left[\gamma(1+\epsilon) - 2\gamma\right] = \frac{\mu\gamma^2}{2\ell^2}\Big[\underbrace{(1+\epsilon)^2 - 2(1+\epsilon)}_{\epsilon^2 - 1}\Big]$$

$$\Rightarrow \boxed{E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)}$$

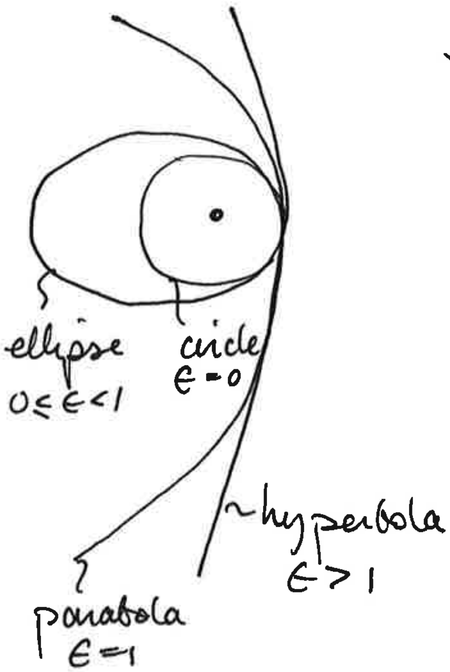
$\epsilon < 1$ (bounded; circle or ellipse)

$\Rightarrow$ $E < 0$ also cond. for bound

$\epsilon \geq 1$ $E \geq 0$ unbounded

! for a given value of ℓ, lowest energy orbit is circular

Unbounded orbits



$$r = \frac{c}{1 + \epsilon \cos\varphi} \qquad E \geq 0,\ \epsilon \geq 1$$

- if $\epsilon \geq 1$, $r(\varphi) \to \infty$ for some φ
- at $\epsilon = 1$, it is for $\varphi = \pm\pi$
  so $r \to \infty$ as $\varphi \to \pm\pi$
  $\Rightarrow$ unbounded – too high T to bind
- cartesian $y^2 = c^2 - 2cx$ for $\epsilon = 1$, parabola
- general $\epsilon > 1$ $\quad \dfrac{(x-\delta)^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$ hyperbola

$$\alpha = \frac{c}{\epsilon^2 - 1} = -a \qquad \beta = \frac{c}{\sqrt{\epsilon^2 - 1}} = \frac{b}{i}$$

$$\delta = \epsilon\alpha = -a\epsilon = -d$$