University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 23: Ch. 8.7-8

19 Oct 2018

1 Two asides

If we go back to our expression for energy with a central, conservative force, we can use the separation of variables methods from Chapter 4 to find an expression for $\dot{r}$:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \tag{1}$$

$$\Longrightarrow \dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{l^2}{\mu^2 r^2}} \tag{2}$$

Using the chain rule, we can note that

$$d\varphi = \frac{d\varphi}{dt}\frac{dt}{dr}dr = \frac{\dot{\varphi}}{\dot{r}}dr \tag{3}$$

which, after using the above substitution, noting $\dot{\varphi} = l^2/\mu r^2$, and integrating, gives us an expression for $\varphi(r)$:

$$\varphi(r) = \int \frac{\pm(l^2/r^2)\,dr}{\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{l^2}{\mu^2 r^2}}} \tag{4}$$

In *principle*, if we have an expression for $U(r)$, we have solved the motion of the orbit. The odds of being able to find a closed-form solution for the integral are not great, but at least numerically we could find the motion given some initial conditions. One thing we can say is that since l is constant, φ monotonically increases or decreases, $\dot{\varphi}$ can't change sign.

One other thing we can do, going back to the relationships from last time is to relate the semi-major and semi-minor axes for a $F = -\gamma/r^2$ force law to energy and angular momentum.

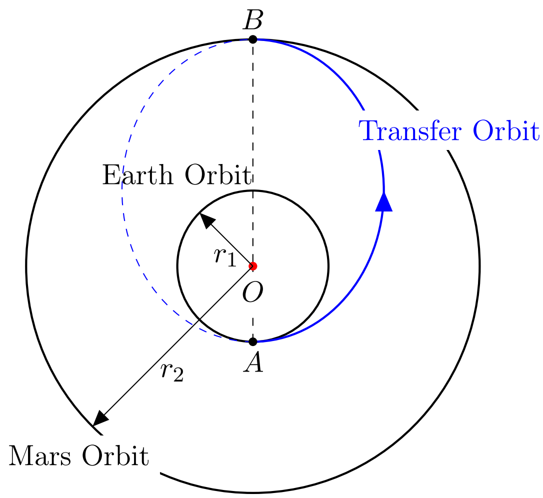
$$a = \frac{c}{1-\epsilon^2} = \frac{\gamma}{2|E|} \qquad b = \frac{c}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \tag{5}$$

Thus the energy of the system determines the semi-major axis, but both energy and angular momentum dictate the semi-minor axis (and thus the eccentricity of the orbit).

2 Orbital dynamics

(A very simplified version.) Traveling to other bodies in the solar system is complicated. We will consider simplified versions of two specific cases: a trip to mars, and a comet/planet flyby.

For a trip to mars, the most economical approach is to move from 1 circular heliocentric orbit to another in the same plane. This is reasonably accurate for the earth-mars system. A method known as the *Hohmann transfer* minimizes the fuel expenditure by transferring from a starting circular orbit to an elliptical orbit that intersects both starting and ending circular orbits. This is shown in the figure below.



Two "burns" are needed to accomplish the transfer. (1) Inject the spacecraft from the circular earth orbit to the elliptical transfer orbit that intersects mars' orbit, and (2) transfer from the elliptical orbit to mars orbit. Clearly timing is key - you have to launch from earth at just the right time (point A) that mars and the spacecraft will both be 180° away at the same time.

We can figure out most of what we need from the energies of the orbit. Previously, we found that the total energy for an orbit is half the potential energy, or $E_{\text{tot}} = -\gamma/2a$. Sitting in earth's orbit at radius r_1, we require

$$E_{\text{tot}} = -\frac{\gamma}{2r_1} = T + U(r_1) = \frac{1}{2}mv_1^2 - \frac{\gamma}{r_1} \tag{6}$$

$$\implies \quad v_1 = \sqrt{\frac{\gamma}{mr_1}} \tag{7}$$

Here m is the mass of the spacecraft. This is the speed required to be in earth's orbit at radius r_1 from the sun. The elliptical transfer orbit needs to hit point A and point B, meaning its semi-major axis has to be half the distance between A and B, $a_t = \frac{1}{2}(r_1 + r_2)$. That lets us determine the energy E_{trans} and velocity v_{t1} required for the transfer orbit at point A. At perihelion (closest to earth, just after the "burn" to get into transfer orbit),

$$E_{\text{trans}} = -\frac{\gamma}{2a_t} = -\frac{\gamma}{r_1 + r_2} == \frac{1}{2}mv_{t1}^2 - \frac{\gamma}{r_1} \tag{8}$$

$$\implies \qquad v_{t1} = \sqrt{\frac{2\gamma}{mr_1}\left(\frac{r_2}{r_1 + r_2}\right)} \tag{9}$$

That means the speed boost required to get into the transfer orbit is

$$\delta v_1 = v_{t1} - v \tag{10}$$

We do the same thing at point B to get from the transfer orbit into the circular orbit at radius r_2, which will give us the speed change required to get into mars' orbit.

$$v_2 = \sqrt{\frac{\gamma}{mr_2}} \qquad \text{speed required in mars' orbit} \tag{11}$$

$$v_{t2} = \sqrt{\frac{2\gamma}{mr_2}\left(\frac{r_1}{r_1 + r_2}\right)} \qquad \text{transfer orbit speed at B} \tag{12}$$

$$\delta v_2 = v_2 - v_{t2} \qquad \text{speed change required to enter mars' orbit} \tag{13}$$

The total speed change required is $\delta v_1 + \delta v_2$. Given a mass, we could use the methods of Chapter 3 to determine the energy required. From Kepler's third law, we can also find the time for transfer, which would be half the orbital period of the elliptical transfer orbit.

$$T_t = \frac{1}{2}\tau_t = \pi\sqrt{\frac{m}{\gamma}}a_t^{3/2} \tag{14}$$

One can transfer more quickly, but it comes at a higher energy cost. So, how about the trip to mars? Plugging in the numbers, the transfer takes 259 days and requires a boost from earth's orbit of $v_{t1} \approx 32.7\,\text{km/s}$ - about three times escape speed! However, that speed is relative to the *sun*, and the earth's orbital velocity is already approximately $30\,\text{km/s}$ - so launch in the direction earth is moving! How about a round trip to mars? Aside from the 259 days of travel, you have to spend 460 days on mars until the alignment is correct again for the Hohmann transfer, making the whole round trip a 2.7 year journey.

3 Flyby

Long story short: we "steal" energy from other bodies. The interaction with another body can speed up a craft with negligible effect on a massive body (e.g., jupiter). We approach the body on a hyperbolic incoming orbit to shoot right past it while gaining speed relative to the sun. The figure below shows a spacecraft approaching and flying behind body B.

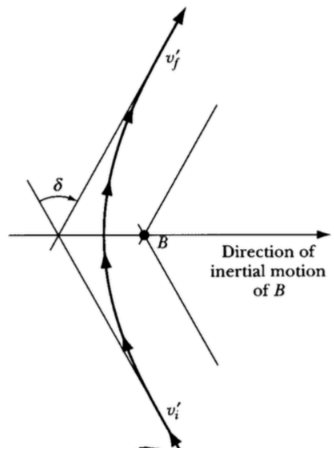


Figure 1: *From Thornton & Marion, "Classical Dynamics".*

The inertial motion of body B relative to the sun is to the right, and the craft approaches body B with initial velocity v_i' relative to B on a straight line trajectory, which will result in its "orbit" around B being hyperbolic. The initial and final velocities *with respect to B* are v_i' and v_f'. The net effect of passing behind B is a deflection in trajectory δ.

Because of the motion of B, the velocities of the spacecraft can be quite different in the sun's reference frame, or another inertial frame in which B's motion occurs. Looking at the velocity addition below, the initial and final velocities of the craft *relative to the sun*, v_i and v_f can be very different due to the motion of B relative to the sun at velocity v_B.

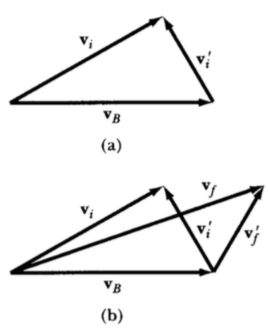
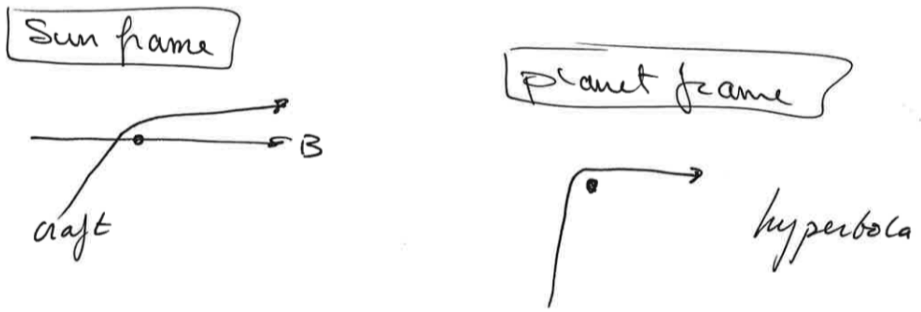


Figure 2: *From Thornton & Marion, "Classical Dynamics".*

We can see $v_i = v_B + v_i'$ and $v_f = v_B + v_f'$. If the craft passes *behind* B, B pulls the craft forward and the final velocity relative to the sun is increased. If the craft passes *in front* of B, B pulls the craft backwards and the final velocity relative to the sun is decreased. Below the path of the craft is shown in a frame of reference moving with the planet, a hyperbolic orbit, and in the sun's frame of reference, which shows the deflection of the craft's trajectory.



The only real problem is that B has to be in the right place at the right time.