University of Alabama
Department of Physics and Astronomy

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Lecture 24: Ch. 11.1-2
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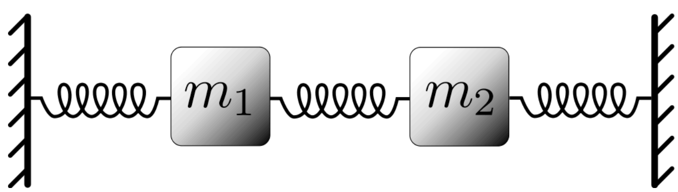
1 Coupled Oscillators

One oscillator (e.g., mass-spring) has one natural frequency. Two or more coupled oscillators will have several natural ("normal") frequencies, and the general motion is a combination (superposition) of vibrations at all the different natural frequencies.

First, we will consider two masses and three springs with no friction, as shown below. Let the springs, from left to right, be k_1, k_2, and k_3. The positions of the masses x_1 and x_2 are measured from their equilibrium positions, and we assume that at equilibrium all three springs are at their relaxed lengths. Spring 2 is what couples the left and right oscillators together - with it present, one mass can't move without the other moving. We can proceed with either a Newtonian or Lagrangian approach. For this problem, neither gives much advantage over the other, so we will stick with a Newtonian approach. For more complex problems, like a double pendulum, the Lagrangian approach has clear advantages.



First, mass 1 feels forces from springs k_1 and k_2. If spring k_2 is stretched by x_2 then it is compressed by x_1, so its total change in length is $\Delta x_2 = x_2 - x_1$. Spring k_1 is only stretched by x_1, so the force equation is easily found.

$$F_1 = -k_1x_1 + k(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2x_2 \tag{1}$$

We can do the same thing for spring 2, and arrive at the (coupled!) equations of motion:

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2 \tag{2}$$
$$m_2\ddot{x}_2 = k_2x_2 - (k_2 + k_3)x_2 \tag{3}$$

Note that these two equations have a compact matrix form (where matrices have a double underline):

$$\underline{\underline{\boldsymbol{M}}}\ddot{\mathbf{x}} = -\underline{\underline{\boldsymbol{K}}}\mathbf{x} \tag{4}$$

Here the definitions are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{labels config. of system; 2 degrees of freedom = 2 elements} \tag{5}$$

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{6}$$

$$\underline{\underline{\boldsymbol{K}}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \qquad \text{off-diagonal elements = coupling!} \tag{7}$$

With only one degree of freedom (1 mass, 1 spring), we are back to our scalar equation $m\ddot{x} = -kx$. Note $\underline{\underline{\boldsymbol{M}}}$ and $\underline{\underline{\boldsymbol{K}}}$ are symmetric - while we won't make much use of that fact now, it is key to the underlying mathematics. How to solve this? A reasonable guess is that there is a stable oscillating solution - there should be solutions where the two carts both oscillate with the same frequency. E.g.,

$$x_{1,2} = \alpha_{1,2} \cos\left(\omega t - \delta_{1,2}\right) \tag{8}$$

$$y_{1,2} = \alpha_{1,2} \sin\left(\omega t - \delta_{1,2}\right) \tag{9}$$

Since we have a linear equation, we can try both of these at once by using a complex exponential as our trial solution and taking the real part at the end.

$$z_1 = x_1(t) + iy_1(t) = \alpha e^{i(\omega t - \delta_1)} = \alpha_1 e^{-i\delta_1 t} e^{i\omega t} = a_1 e^{i\omega t} \tag{10}$$

$$z_2 = a_2 e^{i\omega t} \tag{11}$$

Here we defined $a_1 = \alpha_1 e^{-i\delta_1 t}$ for convenience. Using a complex exponential is nice since it makes the time dependence far simpler, and turns derivatives into multiplication. Combining,

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \mathbf{a} e^{i\omega t} \tag{12}$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{bmatrix} \tag{13}$$

In the end, the actual motion is $\mathbf{x}(t) = \Re[\mathbf{z}(t)]$.[i] Using this form in $\underline{\underline{\boldsymbol{M}}}\ddot{\mathbf{x}} = -\underline{\underline{\boldsymbol{K}}}\mathbf{x}$, taking a second derivative just multiplies the original function by $-\omega^2$:

$$-\omega^2 \underline{\underline{\boldsymbol{M}}}\mathbf{a}e^{i\omega t} = -\underline{\underline{\boldsymbol{K}}}\mathbf{a}e^{i\omega t} \tag{14}$$

$$\text{or} \qquad \left(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}\right)\mathbf{a} = 0 \tag{15}$$

This is a *generalized eigenvalue equation*, a topic we will come back to in later lectures. The above only has non-trivial solutions if the matrix $\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}$ has determinant zero (the trivial solution being $\mathbf{a} = 0$ when nothing moves). Since $\underline{\underline{\boldsymbol{K}}}$ and $\underline{\underline{\boldsymbol{M}}}$ are 2x2 matrices, we'll end up with a quadratic equation giving 2 solutions for 2 normal frequencies.

The general case is rather messy, for now we will assume all k's and m's are the same so we can figure out what the important physics is.

1.1 Identical masses and springs

Now $m_1 = m_2 \equiv m$ and $k_1 = k_2 = k_3 \equiv k$, giving us

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{16}$$

$$\underline{\underline{\boldsymbol{K}}} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \tag{17}$$

and our matrix equation is

$$\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \tag{18}$$

The determinant is

$$\begin{aligned}
\det(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}) &= \left(2k - m\omega^2\right)^2 - k^2 && (19)\\
&= 4k^2 - 4km\omega^2 + m^2\omega^4 - k^2 && (20)\\
&= 3k^2 - 4km\omega^2 + m^2\omega^4 && (21)\\
&= \left(k - m\omega^2\right)\left(3k - m\omega^2\right) && (22)
\end{aligned}$$

Thus, the determinant is zero when $\omega = \omega_1 = \sqrt{k/m}$ and $\omega = \omega_2 = \sqrt{3k/m}$. These are the two (normal) frequencies at which the two carts can oscillate in purely sinusoidal fashion. Note that the

[i] $\Re[x]$ means "take the real part of x"

$\pm$ is not necessary when taking the square root since $+\omega$ and $-\omega$ give the same solution. Sinusoidal motion with any of the normal frequencies is a *normal mode*. The first is just like one cart moving by itself, curious! What is the motion like?

$$\mathbf{x} = \Re[\mathbf{z}] \tag{23}$$

$$\mathbf{z} = \mathbf{a}e^{i\omega t} \tag{24}$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{25}$$

This needs to satisfy $(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}})\mathbf{a} = 0$. Now we know the frequencies, we need to solve for $\mathbf{a}$.

1.2 First normal mode

For the first mode, $\omega_1 = \sqrt{k/m}$, so

$$\underline{\underline{\boldsymbol{K}}} - \omega_1^2 \underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{26}$$

Which has determinant $k^2 - k^2 = 0$ as required. Noting $(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}})\mathbf{a} = 0$,

$$k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \tag{27}$$

This implies $a_1 - a_2 = 0$, and thus $a_1 = a_2 = Ae^{-i\delta}$. The motion is then

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)} \tag{28}$$

$$\mathbf{x}(t) = \Re[\mathbf{z}(t)] = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta) \tag{29}$$

Or, more simply,

$$x_1(t) = A\cos(\omega_1 t - \delta) \tag{30}$$

$$x_2(t) = A\cos(\omega_1 t - \delta) \tag{31}$$

This is the first normal mode, and the masses move with equal amplitudes exactly in phase. That is, they both move in unison and keep a fixed distance apart, so the middle spring stays at its equilibrium length. That means there is no interaction between the two masses, and each cart oscillates as though it were attached to a single spring! That's why the first mode ended up with $\omega_1 = \sqrt{k/m}$ like a single mass-spring system, both masses just oscillate independently since k_2 doesn't change its length.

1.3 Second normal mode

For the second mode, $\omega_2 = \sqrt{3k/m}$, so

$$\underline{\underline{\boldsymbol{K}}} - \omega_1^2 \underline{\underline{\boldsymbol{M}}} = -k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{32}$$

And again we can use $(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}})\mathbf{a} = 0$, yielding

$$-k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \tag{33}$$

Performing the matrix multiplication, we must have $a_1 + a_2 = 0$ or $a_1 = -a_2 = Ae^{i\delta}$. The motion is then

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_2 t} = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)} \tag{34}$$

$$\mathbf{x}(t) = \Re[\mathbf{z}(t)] = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ -A \end{bmatrix} \cos{(\omega_2 t - \delta)} \tag{35}$$

This time, both particles again the same amplitude but they are exactly out of phase! The two masses are either moving toward each other or away from each other. We shouldn't be surprised at how this worked out - there are two degrees of freedom, and really only two basic ways to move: either the masses move in the same direction (first mode) or they move in opposite directions (second mode), any more complicated motion is a superposition of the two. Along those lines, we shouldn't be surprised that the frequency goes as $\sqrt{3k}$ rather than $\sqrt{k}$ – when the outer springs are stretched by some amount, the middle one is stretched twice as much, so each mass in net feels a force three times that of a single spring.

Here's an applet that can help you understand the two modes: https://phet.colorado.edu/en/simulation/legacy/normal-modes – set the polarization to $\Longleftrightarrow$.

1.4 General solution

We have 2 solutions so far, our 2 normal modes:

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) \tag{36}$$

$$\mathbf{x}(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2) \tag{37}$$

Both satisfy $\underline{\underline{\boldsymbol{M}}}\ddot{\mathbf{x}} = -\underline{\underline{\boldsymbol{K}}}\mathbf{x}$. But our differential equations are homogenous, so the sum of these two solutions is also a solution:

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) \tag{38}$$

In fact, this must be the general solution. To start with, we had two 2nd order differential equations, so we expect a total of 4 overall constants to be determined by boundary conditions. In this case, the four are A_1, A_2, δ_1, and δ_2, so the above must be the general solution. The general solution is hard to visualize, but since $\omega_2 = \sqrt{3}\,\omega_1$ we do know from Ch. 5 (see problem 5.17) that the general motion never repeats itself unless since the ratio of the two frequencies is irrational. That is, unless one of the modes has amplitude zero (A_1 or A_2 is zero) and the system is oscillating in one of the normal modes.

1.5 Normal coordinates

In any possible motion of the 2 mass system, both x_1 and x_2 vary in time. In *normal modes*, the motion is sinusoidal, but still *both* vary - you can't move one mass without moving the other. Given the symmetry of the two modes, one would think it is possible to introduced some coordinates that are fixed. For the first mode, where the masses move in unison, we know the difference in the masses position is fixed, whereas in the second mode we know their average position is constant. In fact, that's exactly what we want to do: introduce two new generalized coordinates, one which is constant in the first mode and one which is constant in the second mode.

These *normal coordinates* are less physically transparent than x_1 or x_2, but conveniently they can vary independently of each other - one coordinate only relates to the first mode, and one only relates to the second. Based on our argument above, the two coordinates should be

$$\xi_1 = \frac{1}{2}(x_1 + x_2) \tag{39}$$

$$\xi_2 = \frac{1}{2}(x_1 - x_2) \tag{40}$$

Remember that any two generalized coordinates are fine so long as we can map them back to the original coordinates, this is clearly true of the x's and ξ's. In terms of these normal coordinates, for the first normal mode we have

$$\text{first mode} = \begin{cases} \xi_1(t) = A\cos(\omega_1 t - \delta) \\ \xi_2(t) = 0 \end{cases} \tag{41}$$

This representation makes it clear that in the first mode the *average* position ξ_1 oscillates sinusoidally with frequency ω_1, but the (average) separation of the two masses ξ_2 remains fixed – the masses move in unison. For the second mode.

$$\text{second mode} = \begin{cases} \xi_1(t) = 0 \\ \xi_2(t) = A\cos(\omega_2 t - \delta) \end{cases} \tag{42}$$

It is also now much clearer that in the second normal mode the average separation of the two masses ξ_2 oscillates sinusoidally with the higher frequency ω_2, but their average position remains constant – the masses move symmetrically about the origin.

The general motion of the system is a superposition of ξ_1 and ξ_2, but the nice thing is that ξ_1 oscillates only at frequency ω_1 and ξ_2 oscillates only at frequency ω_2. Each normal coordinate always oscillates at only one of the two normal frequencies, so a normal coordinate specifies the instantaneous displacement of a particular normal mode of the system. If one or the other is zero you are oscillating in a normal mode, otherwise you are oscillating with some mixture of the first and second modes.

In this case the two modes were very simple to interpret physically and mathematically (the masses move together or oppositely), but in more complex systems the use of normal coordinates can greatly simplify things. Each normal coordinate has a characteristic frequency unaffected by the others, and lets us pick out the parts of the system oscillating with normal frequencies ω_i.