ph301 L26
F18

Double pendulum

- now Lagrangian approach is going to be easier
- review 2 carts + 3 springs w/ Lagrangian formalism

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2$$

$$U = \tfrac{1}{2} k_1 x_1^2 + \tfrac{1}{2} k_2 (x_1 - x_2)^2 + \tfrac{1}{2} k_3 x_2^2$$

$$= \tfrac{1}{2}(k_1 + k_2) x_1^2 - k_2 x_1 x_2 + \tfrac{1}{2}(k_2 + k_3) x_2^2$$

$$\mathcal{L} = T - U$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial \mathcal{L}}{\partial x_1} \Rightarrow m_1 \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial \mathcal{L}}{\partial x_2} \Rightarrow m_2 \ddot{x}_2 = k_2 x_1 - (k_2 + k_3) x_2$$

$\Rightarrow \underline{\underline{M}}\ddot{\vec{x}} = -\underline{\underline{K}}\vec{x}$ as before

- no real advantage here - same as Newtonian in the end

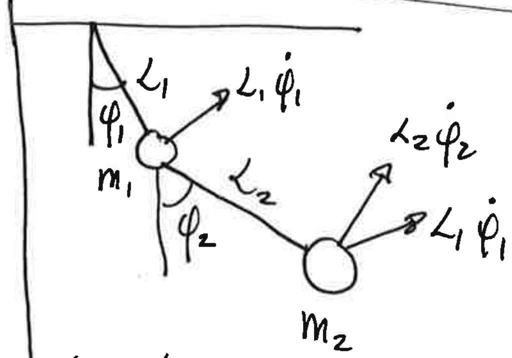
Double Pendulum

Can write down $\mathcal{L}$ easily enough

$$U_1 = m_1 g L_1 (1 - \cos\varphi_1)$$

U_2 - account for m_1 moving too

$$U_2 = m_2 g [L_1(1 - \cos\varphi_1) + L_2(1 - \cos\varphi_2)]$$



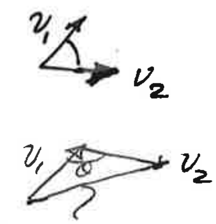
with angular velocities $l\dot\varphi$

$$\Rightarrow U(\varphi_1, \varphi_2) = (m_1 + m_2) g L_1 (1 - \cos\varphi_1) + m_2 g L_2 (1 - \cos\varphi_2)$$

T_1 is easy: $T_1 = \tfrac{1}{2} m_1 (\underbrace{L_1 \dot\varphi_1}_{v = R\omega})^2 = \tfrac{1}{2} m_1 L_1^2 \dot\varphi_1^2$

velocity of m_2? vector sum of relative velocity of 2 + vel. of 1

- $L_2 \dot\varphi_2$ of m_2 rel to its support m_1
- $L_1 \dot\varphi_1$ of its support
- angle between these 2 is $(\varphi_2 - \varphi_1)$



$v_1^2 + v_2^2 + 2 v_1 v_2 \cos\theta$

$$\Rightarrow \quad T_2 = \frac{1}{2} m_2 \big(\underbrace{v_1^2 + v_2^2 + 2 v_1 v_2 \cos(\varphi_2 - \varphi_1)}_{\text{law of cosines}} \big)$$

$$= \frac{1}{2} m_2 \left(L_1^2 \dot\varphi_1^2 + 2 L_1 L_2 \dot\varphi_1 \dot\varphi_2 \cos(\varphi_1 - \varphi_2) + L_2^2 \dot\varphi_2^2 \right)$$

$$\Rightarrow T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot\varphi_1^2 + m_2 L_1 L_2 \dot\varphi_1 \dot\varphi_2 \cos(\varphi_1 - \varphi_2) + \frac{1}{2} m_2 L_2^2 \dot\varphi_2^2$$

- can now write Lagrangian $\mathcal{L} = T - U$ ... complicated!
  - no analytic soln, need an approx, like single pend. (used $\sin\varphi \simeq \varphi$ before)
  - can simulate numerically of course
- presume $\dot\varphi_1$ and $\dot\varphi_2$ and φ_1, φ_2 are small at all times
- keep Taylor exp terms of order 2 or lower

for T: only 2nd term needs help. $\cos \sim 1$ approx:

$$\Rightarrow T \simeq \frac{1}{2}(m_1 + m_2) L_1^2 \dot\varphi_1^2 + m_2 L_1 L_2 \dot\varphi_1 \dot\varphi_2 + \frac{1}{2} m_2 L_2^2 \dot\varphi_2^2$$

- can get away with this since $\cos(\varphi_1 - \varphi_2)$ is already multiplied by doubly small $\dot\varphi_1 \dot\varphi_2$
- have to be more careful w/ U – not the case!

- In $\mathcal{U}$, use $\cos\varphi \simeq 1 - \frac{\varphi^2}{2}$

$$\Rightarrow \quad \mathcal{U} = \tfrac{1}{2}(m_1+m_2)g\,L_1\varphi_1^2 + \tfrac{1}{2}m_2 g L_2 \varphi_2^2$$

- to start: T was transcendental function of φ's and $\dot\varphi$'s
  now: quadratic in $\dot\varphi$'s alone
  $\mathcal{U}$ was transcendental function of φ's
  now: quadratic in φ's
- gain: derivatives of $\mathcal{L}$ will be linear eqns we can solve
  price: only small angles & velocities
  truly odd behavior hidden

- construct $\mathcal{L} = T - \mathcal{U}$, take derivatives

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\varphi_1} = \frac{\partial\mathcal{L}}{\partial\varphi_1} \Rightarrow (m_1+m_2)L_1^2\ddot\varphi_1 + m_2L_1L_2\ddot\varphi_2 = -(m_1+m_2)gL_1\varphi_1$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\varphi_2} = \frac{\partial\mathcal{L}}{\partial\varphi_2} \Rightarrow m_2L_1L_2\ddot\varphi_1 + m_2L_2^2\ddot\varphi_2 = -m_2 g L_2\varphi_2$$

or $\underline{\underline{M}}\,\ddot{\vec\varphi} = -\underline{\underline{K}}\,\vec\varphi$ w/ $\vec\varphi = \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$ matrix eqn

$$\underline{\underline{M}} = \begin{bmatrix}(m_1+m_2)L_1^2 & m_2L_1L_2\\ m_2L_1L_2 & m_2L_2^2\end{bmatrix} \qquad \underline{\underline{K}} = \begin{bmatrix}(m_1+m_2)gL_1 & 0\\ 0 & m_2gL_2\end{bmatrix}$$

- again symmetric matrices
- again off-diagonal terms of matrix = coupling/cross terms
- looks like 2 carts + 3 springs!
  $\underline{\underline{K}}$ isn't a spring but plays analogous role
  gravity = restoring force
  $\underline{\underline{M}}$ isn't mass, but plays role of inertia

- Same eqns have same solns ⇒ try sinusoidal modes

$$\vec{\varphi}(t) = \mathrm{Re}\,\vec{z}(t) \qquad \vec{z}(t) = \vec{a}\, e^{i\omega t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}$$

- as before, must satisfy $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}})\vec{a} = 0$

$$\Rightarrow \det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0 \quad \text{for nontrivial solns}$$

- for now, equal lengths & masses to get at basic behavior

$m_1 = m_2 \equiv m$, $L_1 = L_2 \equiv L$, let $\omega_0 = \sqrt{g/L}$ (single pend freq)

$\Rightarrow g = L\omega_0^2$ to simplify

$$\Rightarrow \underline{\underline{M}} = \underset{\overset{?}{I}!}{mL^2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \underline{\underline{K}} = \underset{\overset{2}{I}!}{mL^2} \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

$$(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = mL^2 \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{bmatrix} \quad \text{set det. to zero} \ldots$$

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0 \quad \text{quadratic in } \omega^2$$

$$\omega^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 8\omega_0^4}}{2} = 2\omega_0^2 \pm \omega_0^2\sqrt{2}$$

$$\Rightarrow \omega_1^2 = (2 - \sqrt{2})\omega_0^2, \quad \omega_2^2 = (2 + \sqrt{2})\omega_0^2 \quad \left(\omega_0 = \sqrt{\tfrac{g}{L}}\right)$$

one mode they move in same dir, one opposing

- from $(\underline{\underline{K}} - \omega^2 M)\vec{a} = 0$, can get the motion

$$@\omega_1, \; (\underline{\underline{K}} - \omega_1^2 \underline{\underline{M}}) = mL^2\omega_0^2(\sqrt{2} - 1) \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

Since $(\underline{\underline{K}} - \omega_1^2 \underline{\underline{M}})\vec{a} = 0$, this implies

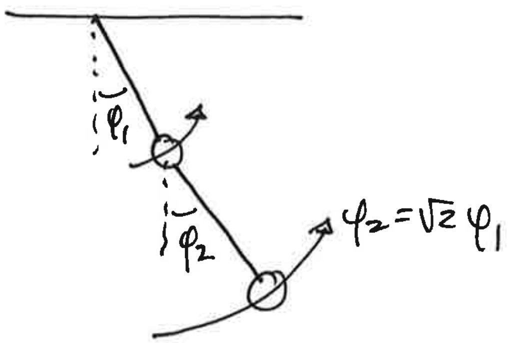
$$\begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 2a_1 - \sqrt{2}a_2 = 0 \quad \Rightarrow a_2 = \sqrt{2}\, a_1$$

1st mode

$$\vec{\varphi}_1(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = Re\left[\vec{a}\, e^{i\omega_1 t}\right] = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1)$$

exactly in phase, lower pendulum has $\sqrt{2}$ ampl of upper

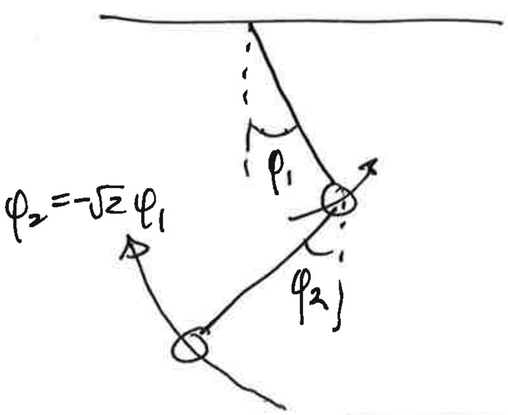
$\boxed{\text{1st mode}}$



second mode? just flip the sign of $\sqrt{2}$ ...

$$\vec{\varphi}_2(t) = A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

φ_2 exactly out of phase w/ φ_1, but still w/ $\sqrt{2}$ amplitude of φ_1



- any general soln is a linear combination of these 2 modes

General Case n degrees of freedom osc about a point of stable equ.

$\Rightarrow$ n generalized coordinates $q_1 \ldots q_n$

$$\vec{q} = (q_1, \cdots, q_n)$$

assumed holonomic ...
num coord = num deg freedom

- for our carts, $\vec{q} = (x_1, x_2)$; double pendulum $\vec{q} = (\varphi_1, \varphi_2)$
- assuming a conservative system $U(q_1, \cdots q_n) = U(\vec{q})$ exists and $\mathcal{L} = T - U$ with $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$

  $\alpha = 1 \cdots N$ particles in the system
- have to rewrite in terms of generalized coord. $\vec{q}$ using their relationship to Cartesian coord $\vec{r}$

  $\vec{r}_\alpha = \vec{r}_\alpha(q_1, \cdots, q_n)$ assuming they are time-independent

now $\dot{\vec{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i \Rightarrow \dot{\vec{r}}_\alpha^{\,2} = \left(\sum_j \frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j \right) \cdot \left(\sum_k \frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_k \right)$

so $T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2} \sum_{j,k} A_{jk} \dot{q}_j \dot{q}_k$

where $A_{jk} = A_{jk}(q_1, \cdots q_n) = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \right)$

are just coefficients that may depend on $\vec{q}$

- Lagrangian has general form $\mathcal{L}(\vec{q}, \dot{\vec{q}}) = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q})$
- presuming small osc about equilibrium at $\vec{q} = 0$ can Taylor expand U

$$U(\vec{q}) = \underbrace{U(0)}_{\text{const, drop it.}} + \underbrace{\sum_j \left. \frac{\partial U}{\partial q_j} \right|_{q=0} q_j}_{\text{zero at equ!}} + \frac{1}{2} \sum_{jk} \boxed{\left. \frac{\partial^2 U}{\partial q_j \partial q_k} \right|_{q=0}} q_j q_k + \cdots$$

~ define K_{jk}

$\Rightarrow U(\vec{q}) = \frac{1}{2} \sum_{jk} K_{jk} q_j q_k$

for T: each term already has $\dot{q}_j \dot{q}_k$, doubly small

$\Rightarrow$ ignore everything but constant factor in $A_{jk}(\vec{q})$

define $A_{jk}(0) = M_{jk}$ $\Rightarrow$ $T(\dot{q}) = \frac{1}{2} \sum_{j,k}' M_{jk} \dot{q}_j \dot{q}_k$

$\Rightarrow$ $\mathcal{L}(\vec{q}, \dot{\vec{q}}) = T(\dot{\vec{q}}) - U(\vec{q})$ same as approx forms for double pendulum !

- reduces T to homogeneous quadratic function of $\dot{q}$'s
  U to homogeneous quadratic function of q's

$\Rightarrow$ equations of motion are solvable linear equations

can arrive at $\underline{\underline{M}} \ddot{\vec{q}} = -\underline{\underline{K}} \vec{q}$ $\quad q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

and $\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0$

just as before, assuming sinusoidal normal modes

next time: equation of motion for n oscillators, examples