for the general case, for small oscillations we could write $\mathcal{L}(q,\dot{q}) = T(\dot{q}) - \mathcal{U}(q)$, $T(\dot{q}) = \frac{1}{2}\sum\limits_{jk}' M_{jk}\dot{q}_j\dot{q}_k$

$$\mathcal{U}(q) = \frac{1}{2}\sum_{jk}' K_{jk} q_j q_k$$

- Since there are n generalized coordinates q_i ($i=1-n$) there are n Lagrange equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q} \quad [i=1\ldots n]$$

- have to differentiate $T, \mathcal{U}$ expressions above
- explicit example for $n=2$

$$\mathcal{U} = \frac{1}{2}\sum_{j,k=1}^{2}{}' K_{jk} q_i q_k = \frac{1}{2}\left(K_{11}q_1^2 + K_{12}q_1q_2 + K_{21}q_2q_1 + K_{22}q_2^2\right)$$

$$\mathcal{U} = \frac{1}{2}\left(K_{11}q_1^2 + 2K_{12}q_1q_2 + K_{22}q_2^2\right) \quad (\text{Since } K_{21}=K_{12})$$

- Can easily differentiate with respect to either q

$$\frac{\partial\mathcal{U}}{\partial q_i} = \sum_j{}' K_{ij} q_j \quad (\text{all } q_i \text{ with } i\neq j \text{ terms are zero!})$$

- KE term works the same way

$$\frac{\partial T}{\partial\dot{q}_i} = \sum_j{}' M_{ij}\dot{q}_j \qquad \frac{d}{dt}\frac{\partial T}{\partial\dot{q}_i} = \sum_j{}' M_{ij}\ddot{q}_j$$

- n Lagrange equations follow

$$\sum_j M_{ij}\ddot{q}_j = -\sum_j K_{ij} q_j \quad \text{or} \quad \underline{\underline{M}}\,\ddot{\vec{q}} = -\underline{\underline{K}}\,\vec{q} \quad w/\ \vec{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

- in other words, it works just like 2 masses + 3 springs
  - suppose stable osc are $\vec{q}(t) = \mathrm{Re}[\vec{z}(t)]$, $\vec{z} = \vec{a}\,e^{i\omega t}$

$\Rightarrow$ Same eigenvalue eqn $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}})\vec{a} = 0$

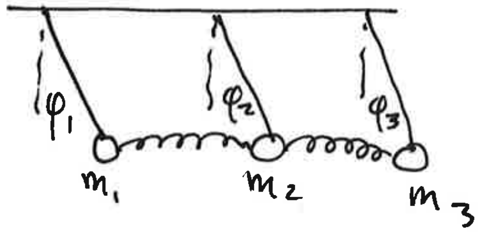
solutions only if ω satisfies characteristic ("secular") eqn

$$\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0$$

$\Rightarrow$ n^{th} degree polynomial for ω^2, with n solutions giving n normal frequencies of the system

as before, generic motion is a superposn of normal modes

3 coupled pendulums



Let $m_1 = m_2 = m_3 \equiv m$

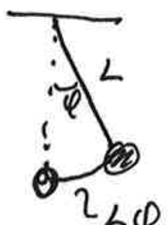
quite tedious; resort to inevitable approximations sooner rather than later

$$T = \tfrac{1}{2} m L^2 \left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 \right)$$ straightforward

- for U_{grav} we know $U_i = mgL(1-\cos\varphi)$, use $\cos\varphi \sim 1 - \frac{\varphi^2}{2}$

$$\Rightarrow \quad U_{grav} \simeq \tfrac{1}{2} mgL \left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \right)$$

- what about the springs? need to know change in length
  quite messy ... but stick to small φ_i



- distance moved $\simeq$ arclength $= L\varphi_i$
- so the relative change for a spring is just $L\Delta\varphi$

$$U_{spr} = \tfrac{1}{2} k L^2 \left[(\varphi_2 - \varphi_1)^2 + (\varphi_3 - \varphi_2)^2 \right]$$

$$= \tfrac{1}{2} k L^2 \left[\varphi_1^2 + 2\varphi_2^2 + \varphi_3^2 - 2\varphi_1\varphi_2 - 2\varphi_2\varphi_3 \right]$$

- Can now construct Lagrangian, but it will be messy
  - typical physicist trick: claim that in "natural" units all annoying constants just happen to be equal to 1
  - you are mostly engineers. do not do this.

Doing it anyway, let $m = L = 1$ (leave g, k alone) so origin of terms is clear

$$\Rightarrow \quad T = \frac{1}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2)$$

$$\mathcal{U} = \frac{1}{2} g(\varphi_1^2 + \varphi_2^2 + \varphi_3^2) + \frac{1}{2} k(\varphi_1^2 + 2\varphi_2^2 + \varphi_3^2 - 2\varphi_1\varphi_2 - 2\varphi_2\varphi_3)$$

No need to bother even writing Lagrangian since we have $\mathcal{L}(q, \dot{q}) = T(\dot{q}) - \mathcal{U}(q)$ again - we will get

$$\underbrace{\underline{\underline{M}}}_{\substack{\text{coeff in } T \\ \text{for } \dot{\varphi}_i}} \ddot{\vec{\varphi}} = -\underbrace{\underline{\underline{K}}}_{\substack{\text{coeff in } \mathcal{U} \\ \text{for } \varphi_i}} \vec{\varphi} \qquad \vec{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{M}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{I_3}} \qquad \underline{\underline{K}} = \begin{bmatrix} g+k & -k & 0 \\ -k & g+2k & -k \\ 0 & -k & g+k \end{bmatrix}$$

- again, presume $\vec{\varphi}(t) = \mathrm{Re}\left[\vec{a}\, e^{i\omega t}\right] \Rightarrow (\underline{\underline{K}} - \omega^2 \underline{\underline{M}})\vec{a} = 0$

requires $\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0$ for nontrivial ω

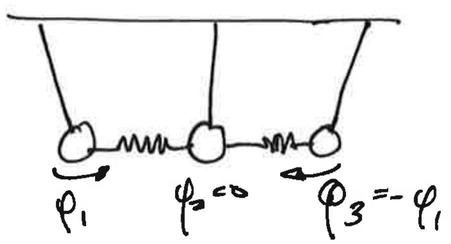
$$(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = \begin{bmatrix} g+k-\omega^2 & -k & 0 \\ -k & g+2k-\omega^2 & -k \\ 0 & -k & g+k-\omega^2 \end{bmatrix}$$

$$\det = (g-\omega^2)(g+k-\omega^2)(g+3k-\omega^2) \quad \text{so} \quad \left\{ \begin{array}{l} \omega_1^2 = g \\ \omega_2^2 = g+k \\ \omega_3^2 = g+3k \end{array} \right\}$$

- substituting each ω back into $(\underline{\underline{K}}-\omega^2\underline{\underline{M}})\vec{a}=0$ we can find amplitudes as before

for 1st mode, $\omega_1=\sqrt{g/L}$, $a_1=a_2=a_3$ — all in phase, move in unison, springs irrelevant

for 2nd mode, $\omega_2=\sqrt{\frac{g}{L}+\frac{k}{m}}$ $a_2=0$, $a_1=-a_3=Ae^{i\delta}$



middle mass stationary, side masses squish it

for 3rd mode, $\omega_3=\sqrt{\frac{g}{L}+\frac{3k}{m}}$ $a_1=-\frac{1}{2}a_2=a_3=Ae^{i\delta}$

outer pendulums oscillate in unison, middle one osc @ twice the amplitude and out of phase, it pinballs back & forth