ph301 F18
L29 ch9

on to rotating frames.

Angular velocity vector

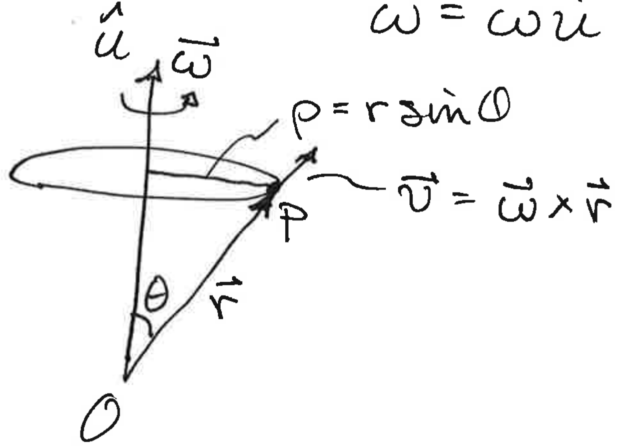
usually there is one fixed point of interest

- axle, pivot
- if not, use CM - find motion of CM, then analyze rotation relative to the CM

Euler's theorem - general motion of any body relative to a fixed point O is a rotation about some axis through O

$\Rightarrow$ just need direction of this axis and angle of rotation

- we worry mainly about rate of rotation - angular vel.
- direction of axis of rotation is a unit vector $\hat{u}$
  rate is $\omega = d\theta/dt$
  - e.g. merry-go-round: $\hat{u}$ is vertical
- can use this to form an angular velocity vector
  $\vec{\omega} = \omega\hat{u}$ orientation by right hand rule



- rotating particle is on a circular path with radius $\rho = r\sin\theta$
- moves with $v = \omega\rho = \omega r\sin\theta$
- accounting for direction
  $\vec{v} = \vec{\omega}\times\vec{r} = \omega(\hat{u}\times\vec{r})$
- generalization of $v = r\omega$

- similar relationship for *any* vector fixed in the rotating body
  if $\hat{e}$ is a unit vector fixed in the body, then

$$\frac{d\hat{e}}{dt} = \vec{\omega} \times \hat{e} \quad \text{useful!}$$

Adding angular velocities

- frames 2 & 1 have relative velocity $\vec{v}_{21}$, and body 3 has velocity $\vec{v}_{32}$ relative to frame 2, then 3 relative to 1 is

$$\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21} \quad \text{usual velocity addition}$$

- Say frame 2 rotates with $\vec{\omega}_{21}$ relative to frame 1 and both frames 1 & 2 have the same origin O and 3 rotates about O w/ $\vec{\omega}_{31}$ and $\vec{\omega}_{32}$ relative to frames 1 & 2. Since $\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21}$, must be that

$$\vec{\omega}_{31} \times \vec{r} = (\vec{\omega}_{32} \times \vec{r}) + (\vec{\omega}_{21} \times \vec{r}) = (\vec{\omega}_{32} + \vec{\omega}_{21}) \times \vec{r}$$

true for any $\vec{r}$ $\Rightarrow$ $\vec{\omega}_{31} = \vec{\omega}_{32} + \vec{\omega}_{21}$

angular velocities add like regular velocities

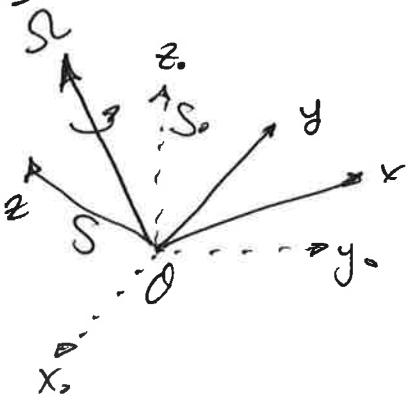
Notation $\vec{\Omega}$ = angular velocity of a non-inertial, rotating reference frame relative to which we are calculating motion of object(s)

(like $\vec{A}$, $\vec{V}$ last time for accel, vel of frames)

- e.g. $\vec{\Omega}$ = angular velocity of earth
- $\vec{\omega}$ usually an unknown

Time derivatives in a rotating frame

e.g. earth: $\Omega = \dfrac{2\pi \text{ rad}}{24 \times 60 \times 60 \text{ s}} \simeq 7.3 \times 10^{-5}$ rad/s — small, so we can usually ignore



- 2 frames S_0, S share origin O
- only relative motion is rotation at $\vec{\Omega}$

  e.g. S = earth, O = earth's center
  S_0 = same origin but fixed to distant stars

- Consider any arbitrary vector $\vec{Q}$ (pos/vel/force/whatever)
  rate of change of $\vec{Q}$ in S vs S_0?

  $\left(\dfrac{d\vec{Q}}{dt}\right)_{S_0}$ = rate of change of $\vec{Q}$ relative to inertial frame S_0

  $\left(\dfrac{d\vec{Q}}{dt}\right)_{S}$ = rate of change of $\vec{Q}$ relative to rotating frame S

- expand $\vec{Q}$ in terms of 3 orthonormal unit vectors
  $\hat{e}_1, \hat{e}_2, \hat{e}_3$ that are fixed in the rotating frame S
  (could be x, y, z)

$$\vec{Q} = Q_1 \hat{e}_1 + Q_2 \hat{e}_2 + Q_3 \hat{e}_3 = \sum_{i=1}^{3} Q_i \hat{e}_i$$

- convenient for observers in S, but valid in either frame
  (but in S_0 observers see the $\hat{e}_i$ as rotating while in S they appear fixed.)

time variation? $\left(\frac{d\vec{Q}}{dt}\right)_S = \sum_i' \underbrace{\frac{dQ_i}{dt}}\hat{e}_i$ since $\hat{e}_i$ are fixed in S

expansion is the same in either frame

in S_0, $\hat{e}_i$ are not fixed $\Rightarrow$ $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \underbrace{\sum_i' \frac{dQ_i}{dt}\hat{e}_i}_{\dot{Q}_S\,!} + \sum_i' Q_i \underbrace{\left(\frac{d\hat{e}_i}{dt}\right)_{S_0}}_{\text{not fixed in } S_0!}$

- in S_0, $\hat{e}_i$ is rotating. using prev result,

$$\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \vec{\Omega}\times\hat{e}_i$$

$$\Rightarrow \sum_i' Q_i\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \sum_i' Q_i(\vec{\Omega}\times\hat{e}_i) = \vec{\Omega}\times\underbrace{\sum_i' Q_i\hat{e}_i}_{\text{same in both frames!}} = \vec{\Omega}\times\vec{Q}$$

$$\Rightarrow \left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{dQ}{dt}\right)_S + \vec{\Omega}\times\vec{Q}$$ will let us get Newton's law in the rotating frame S!

Newton's 2nd law in a rotating frame

- assume angular velocity $\vec{\Omega}$ of S relative to S_0 is constant
  - if this is true in one frame it is true in all

$$\left(\frac{d\vec{\Omega}}{dt}\right)_{S_0} = \left(\frac{d\vec{\Omega}}{dt}\right)_S + \vec{\Omega}\times\vec{\Omega} = \left(\frac{d\Omega}{dt}\right)_S \checkmark$$

- consider particle of mass m at $\vec{r}$. In inertial frame S_0,

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$$

using our result, $\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$

differentiate: $\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$

use our result again

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_S\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

to clean up, let $\dot{\vec{Q}} = \left(\frac{d\vec{Q}}{dt}\right)_S$, and note $\Omega = \text{const}$

$$\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \ddot{\vec{r}} + \left(2\vec{\Omega}\times\dot{\vec{r}}\right) + \left(\vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right)$$

dots $= \frac{d}{dt}$ in S

plug back into Newton's law:

$$m\ddot{\vec{r}} = \vec{F} + \underbrace{\left(2m\dot{\vec{r}}\times\vec{\Omega}\right)}_{\text{Coriolis force}} + \underbrace{\left[m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}\right]}_{\text{Centrifugal force}}$$

or $m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}$

Centrifugal force proportional to $\vec{v} = \dot{\vec{r}}$ relative to rotating frame

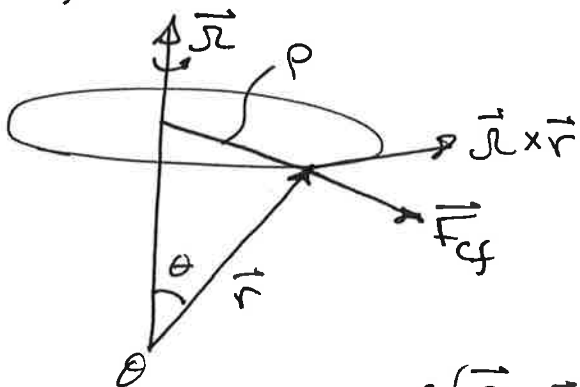
$\Rightarrow$ zero for any object at rest in rotating frame, negligible for slow objects

order of mag: $F_{cor} \sim mv\Omega$, $F_{cf} \sim mr\Omega^2 \Rightarrow \frac{F_{cor}}{F_{cf}} \sim \frac{v}{R_E\Omega} \sim \frac{v}{V}$

here V = speed of a point on the equator, and $r \sim R_e$ near surface

$V \sim 1000$ mph, so for projectiles with $v << 1000$ mph we can ignore the Coriolis force – do this for now

$$\vec{F}_{cf} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$



- earth's rotation carries object around a circle of latitude $\vec{\Omega} \times \vec{r}$ is tangential to circle = velocity of circular motion as seen from non-rotating frame
- $(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$ is then radially outward along $\hat{\rho}$, magnitude $\Omega^2 r \sin\theta = \Omega^2 \rho$

$\Rightarrow \quad \vec{F}_{cf} = m\Omega^2 \rho \hat{\rho}$ for observers rotating with earth, there is a centrifugal force radially outward from earth's axis

– if we let $\vec{v} = \vec{\Omega} \times \vec{r}$ = velocity associated w/ earth's rotation (seen from non-rotating frame), then $v = \Omega \rho$ and $F_{cf} = mv^2/\rho$ as usual!

$\boxed{\text{next time}}$: makes free fall accel surprisingly complex!

$$\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta \hat{\rho}$$
?

- largest at equator ($\theta = \pi/2$) $\sim 0.034 \frac{m}{s^2} \sim 0.3\%$
- zero at poles ($\theta = 0, \pi$)
- difference is small but measureable