University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 2

24 Aug 2018

1 Warmup: HW1 problem 3

1. **(a)** Prove that if $\mathbf{v}(t)$ is any vector that depends on time but which has constant *magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. **(b)** Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\mathbf{r}/dt$ has to be in the direction of $\hat{\boldsymbol{\varphi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

Solution: **(a)** If $|\mathbf{v}|$ is constant, then so is $\mathbf{v}^2$, and its time derivative must be zero.

$$\frac{d\mathbf{v}^2}{dt} = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{1}$$

Since the dot product is commutative, $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$,

$$\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 2\dot{\mathbf{v}}\cdot\mathbf{v} = 0 \tag{2}$$

$$\implies \quad \dot{\mathbf{v}}\cdot\mathbf{v} = 0 \tag{3}$$

Hence, $\mathbf{v}$ and $\dot{\mathbf{v}}$ must be orthogonal.

(b) Just work backwards. We know $\mathbf{v}\cdot\dot{\mathbf{v}} = 0$. Proceeding,

$$\mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{4}$$

$$2\mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{5}$$

$$2\mathbf{v}\cdot\dot{\mathbf{v}} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{6}$$

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d\mathbf{v}^2}{dt} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{7}$$

$$\implies \quad \mathbf{v}^2 = \text{const} \tag{8}$$

Since $\mathbf{v}^2 = |\mathbf{v}|^2$ is constant, clearly $|\mathbf{v}|$ is constant.

2 2D polar coordinates

Figure 1: *Definition of polar coordinates r and ϕ. From "Classical Mechanics" by Taylor, Fig. 1.10*

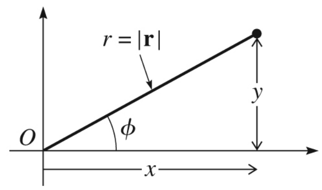


Figure 1.10
Taylor CLASSICAL MECHANICS

Studying rotation in $x-y$ coordinates can be stupid and messy. Why not adopt coordinates that have the same symmetry as the problem? Rectangular problem, rectangular coordinates. Circular problem, circular coordinates.

Advantage: using (r,ϕ) for circular motion has only ϕ changing in time, far less messy math. As defined above:

- ϕ is the angle in the $x-y$ plane relative to the $+x$ axis.
- r is the displacement from the origin

Clearly,

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \qquad \Longleftrightarrow \qquad \left\{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\tfrac{y}{x} \quad \text{careful with quadrant of } \phi \end{aligned}\right. \tag{9}$$

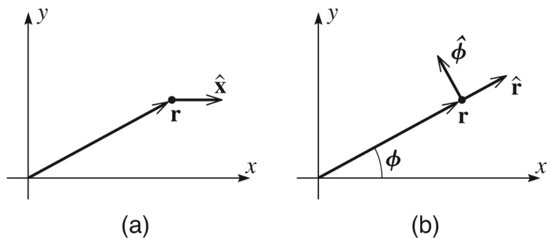
We still need unit vectors. As we discussed last time, they need to be orthogonal (really orthonormal since they must also have magnitude 1).[i]

- We are thus choosing an **orthonormal basis**, which will make all members of the basis (unit vectors) linearly independent
- this means *any* vector can be constructed out of the basis vectors (unit vectors)
- this does **not** work if the basis is not orthonormal (unit vectors not perpendicular and unit magnitude)

How to define? $\hat{\mathbf{x}}$ means move 1 unit in direction of increasing x with y held constant. Can define $\hat{\mathbf{r}}$ analogously (fig **(a)** below):

[i] See http://graphics.ics.uci.edu/ICS6N/NewLectures/Lecture10,11.pdf for example.

Figure 2: *Defining polar unit vectors. From "Classical Mechanics" by Taylor, Fig. 1.11*



$\hat{\mathbf{r}}$ means move 1 unit in the direction of increasing r with ϕ fixed. So $\hat{\mathbf{r}}$ is in the $\mathbf{r}$ direction, but with magnitude 1! This points to a simple definition:

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} \tag{10}$$

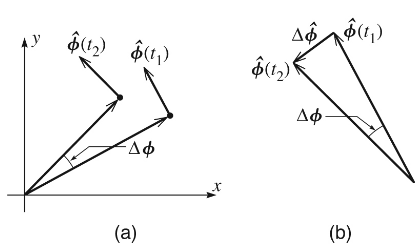
This works in general - by construction it is a vector of unit magnitude in the direction of the original vector! This still doesn't make the $\hat{\boldsymbol{\varphi}}$ construction obvious, however. See figure **(b)** above.

Properties of $\hat{\boldsymbol{\varphi}}$ we require:

- along $+\phi$ direction if $\mathbf{r}$ is fixed
- need an orthonormal basis, meaning
  1. normalized (unit magnitude) $|\hat{\boldsymbol{\varphi}}| = 1$
  2. orthogonal to $\hat{\mathbf{r}}$: $\hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{r}} = 0$

By drawing two different r vectors with different ϕ, you can easily see that both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ change as the position vector $\mathbf{r}$ moves, very much unlike $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$.

Figure 3: *The direction of $\hat{\boldsymbol{\varphi}}$ depends on ϕ. Polar unit vectors are a bit more slippery than their cartesian counterparts. From "Classical Mechanics" by Taylor, Fig. 1.13*



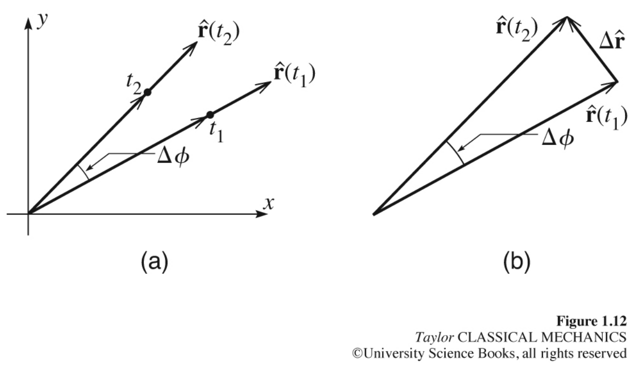
Presuming we can figure out what $\hat{\boldsymbol{\varphi}}$ is so that $\forall\,\hat{\boldsymbol{\varphi}} : \hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{r}} = 0$, we would do something like $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\phi\hat{\boldsymbol{\varphi}}$ as we would with any orthonormal basis of unit vectors like $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$. By the way, $\forall$ means "for all" or "for every."

Say we were describing a pendulum, a mass on a spring. Then the components have physical meaning. F_r is the tension, and F_ϕ is the air resistance along the circular path of the mass, usually ignored.

Position vectors are easy in polars - it is just $\mathbf{r} = r\hat{\mathbf{r}}$. The next step is trickier: how about Newton's laws in cartesian coordinates? We did $\mathbf{F} = m\ddot{\mathbf{r}}$, giving 3 equations for each coordinate, $F_x = m\ddot{x}$, etc. BUT. In doing that, we relied on the obvious fact that $\dot{\hat{\mathbf{x}}}$ is obviously ZERO. What about differentiating $\mathbf{r} = r\hat{\mathbf{r}}$? *Both* r and $\hat{\mathbf{r}}$ can change in time now. Same for ϕ. So we'll need the time derivatives of the polar unit vectors.

3 Time variation of polar unit vectors

Figure 4: *Position of a particle at two successive instants. From "Classical Mechanics" by Taylor, Fig. 1.12*



Consider a moving particle. So long as the particle is not moving purely radially, the unit vector $\hat{\mathbf{r}}(t)$ points in different directions at different times, *and* so does $\hat{\boldsymbol{\varphi}}$. So if we want the radial velocity, we need

$$\dot{\mathbf{r}} = \frac{d}{dt} r\hat{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}} \tag{11}$$

We need the time derivative of $\hat{\mathbf{r}}$ to proceed. From the right hand figure above, for small Δt, we can say

$$\Delta\phi = \phi_2 - \phi_1 \approx \dot{\phi}\Delta t \qquad \text{short times, linearly changing angle} \tag{12}$$

$$\Delta\hat{\mathbf{r}} = |\hat{\mathbf{r}}|\Delta\phi\,\hat{\boldsymbol{\varphi}} = \Delta\phi\,\hat{\boldsymbol{\varphi}} \approx \dot{\phi}\Delta t\,\hat{\boldsymbol{\varphi}} \qquad \text{arclength formula} \tag{13}$$

Note that this means $\Delta\hat{\mathbf{r}}$ is perpendicular to $\hat{\mathbf{r}}$ because it points in the $\hat{\boldsymbol{\varphi}}$ direction. Divide both sides by Δt and let $\Delta t \to 0$, and we have

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\hat{\mathbf{r}}} = \dot{\phi}\,\hat{\boldsymbol{\varphi}} \tag{14}$$

So $d\hat{\mathbf{r}}/dt$ points along the $\hat{\boldsymbol{\varphi}}$ direction, and is proportional to the rate of change of ϕ, neither of which is surprising. In some sense $d\hat{\mathbf{r}}/dt$ is telling you how much you're turning, which is along $\hat{\boldsymbol{\varphi}}$. We don't know that $\dot{\hat{\boldsymbol{\varphi}}}$ is yet, but we can make some progress anyway.

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}} \tag{15}$$

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\varphi}} \tag{16}$$

We can immediately identify the polar components of velocity:

$$v_r = \dot{r} \tag{17}$$

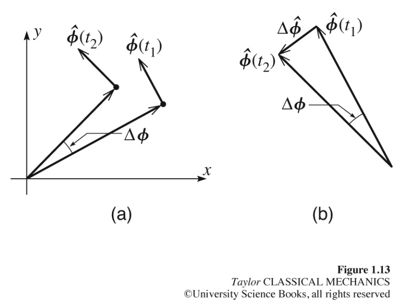
$$v_\phi = r\dot{\phi} \equiv r\omega \tag{18}$$

Just like we remember. Note $\equiv$ means "equal by definition." We have to differentiate one more time to get acceleration.

$$\mathbf{a} \equiv \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}\left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\varphi}}\right) \tag{19}$$

We need the derivative of $\hat{\boldsymbol{\varphi}}$ to get any farther. From the figure below:

Figure 5: *Position of a particle at two successive instants. From "Classical Mechanics" by Taylor, Fig. 1.12*



we can go through the same procedure we did to find $\dot{\hat{\mathbf{r}}}$, and you should convince yourself

$$\dot{\hat{\boldsymbol{\varphi}}} = \frac{d\hat{\boldsymbol{\varphi}}}{dt} = -\dot{\phi}\,\hat{\mathbf{r}} \qquad \text{compare to} \qquad \dot{\hat{\mathbf{r}}} = \frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\,\hat{\boldsymbol{\varphi}} \tag{20}$$

Now we can figure out the acceleration.

$$\mathbf{a} \equiv \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}\left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\varphi}}\right) \quad \text{chain rule} \ldots \tag{21}$$

$$\mathbf{a} = \left(\ddot{r}\,\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt}\right) + \left(\left(\dot{r}\dot{\phi} + r\ddot{\phi}\right)\,\hat{\boldsymbol{\varphi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\varphi}}}{dt}\right) \tag{22}$$

$$\text{substituting unit vector derivatives and simplifying} \ldots \tag{23}$$

$$\mathbf{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\varphi}} \tag{24}$$

Obviously this is horrifying. As we should have expected when we let both r and ϕ vary all willy-nilly. Let's check special cases. Let $r = \text{const}$, so $\dot{r} = \ddot{r} = 0$:

$$\mathbf{a} = -r\dot{\phi}^2\,\hat{\mathbf{r}} + r\ddot{\phi}\,\hat{\boldsymbol{\varphi}} \tag{25}$$

$$\text{or} \quad \mathbf{a} = -r\omega^2\,\hat{\mathbf{r}} + r\alpha\,\hat{\boldsymbol{\varphi}} \tag{26}$$

Here $\omega = \dot{\phi}$ and $\alpha = \ddot{\phi}$. This is our result from intro physics: the radial part has a centripetal acceleration of $r\omega^2 = v^2/r$, and a tangential acceleration $r\alpha$.

What about the other terms?

$$\ddot{r}\,\hat{\mathbf{r}} = (\text{radial acceleration})\,\hat{\mathbf{r}} \qquad \text{makes sense} \tag{27}$$

$$2\dot{r}\dot{\phi}\,\hat{\boldsymbol{\varphi}} \quad \text{Coriolis, due to rotating reference frame (earth!). Ch. 9} \tag{28}$$

So polars look a bit horrifying, but not as horrifying as trying to handle rotation in cartesian coordinates.

Now we can write Newton's laws in polar coordinates:

$$\mathbf{F} = m\mathbf{a} \qquad \Longleftrightarrow \qquad \begin{cases} F_r = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases} \tag{29}$$

Not nearly so simple as cartesian coordinates! This is one reason for redoing Newtonian mechanics in the Lagrangian formalism: why should the coordinates matter if the physical vectors are independent of their coordinate description? The vector $\mathbf{r}$ is the same arrow no matter how we describe it. We will find new methods that are just as easy in either coordinate system based on energy - where we really don't need vectors anyway!

But for now, why would we bother with this complicated nonsense? Because if the coordinates match the symmetry of the problem, it might still end up being simpler.

4 Example problem easier in polars

Skateboard on a frictionless half pipe of radius R. If I release the board near the bottom, what is the subsequent motion?

Figure 6: *Skateboard on a frictionless half pipe of radius R. From "Classical Mechanics" by Taylor, Fig. 1.12*

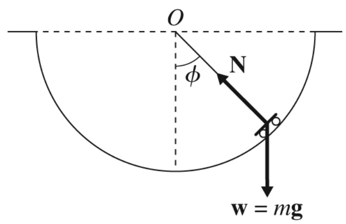


Figure 1.14
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The origin is at the center of the semicircle, and clearly $r = R$ is constrained to be constant. In the case $\dot{r} = 0$,

$$F_r = -mR\dot{\phi}^2 \tag{30}$$

$$F_\phi = mR\ddot{\phi} \tag{31}$$

The forces we have are the normal force $\mathbf{N}$ and the weight mg. We've done this before:

$$F_r = mg\cos\phi - N \tag{32}$$

$$F_\phi = -mg\sin\phi \tag{33}$$

Let's look at the radial equation first. Equating the last two radial equations:

$$mg\cos\phi - N = -mR\dot{\phi}^2 \tag{34}$$

BUT! If R is constant, do we even care about N or the radial force? The particle has to stay on the track anyway, so whatever the radial forces are the radial motion is obvious: $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = 0$. The angular dynamics must be the important part. Check it:

$$-mg\sin\phi = mR\ddot{\phi} \tag{35}$$

$$\ddot{\phi} = -\frac{g}{R}\sin\phi \tag{36}$$

This particular differential equation has no analytic solution (you'll solve it numerically for the HW for bonus points). However, it is rather simple looking, it involves only one coordinate since R is constant, and should be readily amenable to numerical methods. The same would *not* be true if we solved the problem in cartesian coordinates - we would end up with two coupled differential equations, and that is a real pain.

What can we say without much work?

- If $\phi = 0$ then $\ddot{\phi} = 0$, so if we place the particle at rest ($\dot{\phi} = 0$) at the origin, it will just sit there in a stable equilibrium. Duh.
- Suppose we displace to the right by some angle $+\delta\phi$. Then $\ddot{\phi} < 0$ and the resulting acceleration is to the left - a restoring force!
- If ϕ is small, $\sin\phi \approx \phi$ and we have $\ddot{\phi} = -\frac{g}{R}\phi$ – simple harmonic motion (SHM) with $\omega = \sqrt{g/R}$. This is *not* obvious if you solve it in cartesian coordinates!

The same equations have the same solutions! If a reasonable and well-defined approximation like $\phi << 0.5$ (good to 5%) can make it look like SHM, that is a valuable special case! ECE - you'll do this with circuits. The SHM approximation makes it look just like an ideal pendulum with a string of length R, which is a thing we understand.

What is the SHM solution? Both $\sin\omega t$ and $\cos\omega t$ work. But we have a 2nd order differential equation, which requires 2 boundary conditions. There are two ways to do this.

- linear differential equations allow linear superposition of particular solutions, e.g.,

$$\phi(t) = A\cos(\omega t) + B\sin(\omega t) \tag{37}$$

- We can rewrite the above with trig substitutions into a more convenient form:

$$\phi(t) = C\cos(\omega t + \delta) \tag{38}$$

- the latter form gives the constants C and δ obvious physical meanings

Both are equivalent, and since they solve the equation and have the required number of boundary conditions, this is the complete solution. Any solution of an equation like $\ddot{\phi} = -\omega^2 x$ ($\omega \in \mathbb{R}$) can be written in these two forms, which is why we "massage" equations into this form fairly often.

$0.02 version: if you an make an equation look like $\ddot{\phi} = -\omega^2 x$, either exactly or in a well-defined approximation, the solutions are as above regardless of the origin of the equation. **Physically:** the skateboard oscillates sinusoidally about the origin for small displacements. The period depends on the strength of gravity - which acts as the restoring force - and the curvature ($1/R$). Shallower curvature, lower frequency.

5 Ending

Review your vector math - dot and cross product, unit vectors, orthogonal coordinates (in 2D for now).

Let's look at HW with the time remaining.