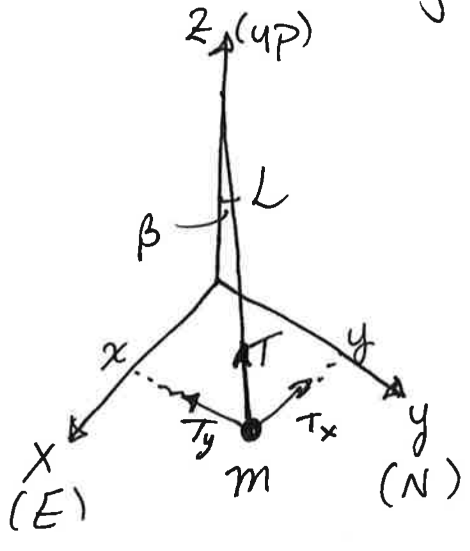
Foucault Pendulum

- very heavy mass m, very long cord
- in an inertial frame, only 2 forces: tension $\vec{T}$, weight $m\vec{g}_0$
- rotating frame: centrifugal and Coriolis as well

$$m\ddot{\vec{r}} = \vec{T} + \underbrace{m\vec{g}_0 + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}_{\text{combine to give } m\vec{g} \text{ as before}} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

let x = east, y = north, z = up



- look at small osc, β small

$$T_z = T\cos\beta \simeq T$$

if vertical $T = mg$, so small β

$$T_z \simeq mg$$

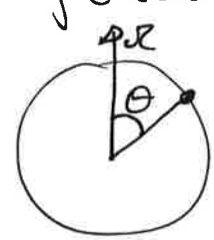
- $\dfrac{T_x}{T} = \dfrac{-x}{L}$, $\dfrac{T_y}{T} = \dfrac{-y}{L}$ (similar triangles)

$$\Rightarrow T_x = -\frac{mgx}{L}, \quad T_y = -\frac{mgy}{L}$$

plug back in, ignore $\dot{z}$ (tiny)

$$\Rightarrow \ddot{x} = -\frac{gx}{L} + 2\dot{y}\,\Omega\cos\theta$$

$$\ddot{y} = -\frac{gy}{L} - 2\dot{x}\,\Omega\cos\theta$$



$\dfrac{g}{L} = \omega_0^2$ natural frequency

$\Omega\cos\theta = \Omega_z$ z component of earth's angular velocity use our previous trick (Ch 2) to solve coupled eqns

let $\eta = x + iy$

multiply y eqn by i and add to x eqn

$$\ddot{x} - 2\Omega_z \dot{y} + \omega_0^2 x = 0$$
$$\ddot{y} + 2\Omega_z \dot{x} + \omega_0^2 y = 0$$
$$\Downarrow$$
$$\ddot{\eta} + 2i\Omega_z \dot{\eta} + \omega_0^2 \eta = 0$$

we've seen this one.
try $\eta = e^{-i\alpha t}$

$$\Rightarrow \quad \alpha^2 - 2\Omega_z \alpha - \omega_0^2 = 0 \quad \Rightarrow \quad \alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} \simeq \Omega_z \pm \omega_0$$

$(\omega_0 \gg \Omega_z)$

2 indep solns as required; general soln is linear combo.

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

fix C_1, C_2 with initial conditions

at $t=0$, pulled to $x = A, y = 0$ with $v_{x0} = v_{y0} = 0$

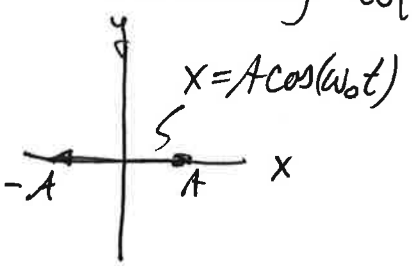
$$\Rightarrow C_1 = C_2 = \frac{A}{2}$$

$$\Rightarrow \eta = x(t) + iy(t) = Ae^{-i\Omega_z t} \cos\omega_0 t$$

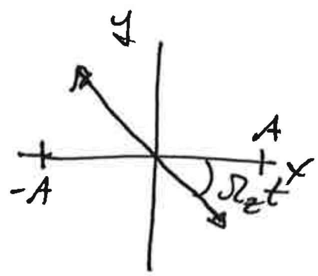
- Since $\omega_0 \gg \Omega_z$, $\cos(\omega_0 t)$ oscillates many times before complex exponential changes much
- initially x oscillates as $\cos(\omega_0 t)$ – SHM along x
  $y \simeq 0$

- eventually $e^{-i\Omega_z t}$ changes, rotating complex number η
  - mixes x, y amplitude while overall amplitude const
  - this rotates the direction of oscillation

- in the northern hemisphere $\Omega_z > 0$, η oscillates sinusoidally w/ $\cos(\omega_0 t)$ but rotates clockwise @ Ω_z



$t \simeq 0$



$t > 0$

- at colatitude θ (latitude $90^\circ - \theta$), rate at which plane of oscillation rotates is $\Omega_z = \Omega \cos\theta$

- at the poles, rotates in a fixed plane at same rate as earth rot (so 1 full rotation per day)
- at the equator, $\theta = 90^\circ$, does not rotate at all

- here, $90^\circ - \theta \simeq 33^\circ$ so $\cos\theta \simeq \frac{1}{2}$, $\Omega_z \simeq \frac{1}{2}\Omega$

  $\Omega = \frac{360^\circ}{\text{day}}$, $\frac{1}{2}\Omega = \frac{180^\circ}{\text{day}} \sim 45^\circ$ in 6 hrs - easy to see

  ? reasonable for a big pendulum constructed well

Coriolis force & accel

Newton's 2nd in 2D polars (ch 1), for a particle

$$\vec{F} = m\ddot{\vec{r}} \iff \begin{array}{l} F_r = m(\ddot{r} - r\dot{\varphi}^2) \\ F_\varphi = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \end{array}$$

Compare to a co-rotating frame

- inertial frame S with origin O
- non-inertial frame S' w/ same origin, rotating w/ const Ω
  - chosen such that $\Omega = \dot{\varphi}$ at $t = t_0$
    i.e. particle and S' rotate at the same rate at t_0
- particle at (r', φ') relative to S'. Since origins are the same
  $r' = r$
- Since rotation rate is the same, $\dot{\varphi}' = 0$ at $t = t_0$
- in S', Newton's Law w/ centrifugal & Coriolis:

$$\vec{F} + \vec{F}_{cf} + \vec{F}_{cor} = m\ddot{\vec{r}}' \quad \text{rewrite in polars}$$

- $r = r'$, $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\Omega}$, $\vec{v}'$ purely radial in co-rotating frame
- $\Rightarrow \vec{F}_{cor}$ in φ' dir, φ' component $-2m\dot{r}\Omega$
- $\ddot{\vec{r}}'$ as in 2D polars, but $\dot{\varphi}' = 0$ at t_0
  so $\dot{\varphi}$ term is absent

$\Rightarrow$

$$\Rightarrow \quad \vec{F} + \vec{F}_{cf} + \vec{F}_{cor} = m\ddot{\vec{r}}\,' \quad \Longleftrightarrow \quad \begin{aligned} F_r + mr\Omega^2 &= m\ddot{r} \\ F_\varphi - 2m\dot{r}\Omega &= mr\ddot{\varphi} \end{aligned}$$

(S' rotates at constant rate, so $\ddot{\varphi} = \ddot{\varphi}\,'$)

Compare to 2D polar:
$$F_r + mr\dot{\varphi}^2 = m\ddot{r} \quad \text{or} \quad F_r = m(\ddot{r} - r\dot{\varphi}^2)$$
$$F_\varphi - 2m\dot{r}\dot{\varphi} = mr\ddot{\varphi} \quad \text{or} \quad F_\varphi = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})$$

- because $\dot{\varphi} = \Omega$, exactly the same eqns
- in non-rotating frame, only terms on left are real net forces F_r, F_φ. On right side, we have centripetal ($-r\dot{\varphi}^2$) and Coriolis ($2\dot{r}\dot{\varphi}$) accelerations
- in rotating frame, don't have either of these accels but they show up as centrifugal ($m\Omega^2 r$) force and Coriolis ($-2m\dot{r}\Omega$) force
- inertial frame forces are simpler – no fake forces – but accelerations are more complex
- rotating frame: forces complex, accel. simple
  - but same eqns, equally valid
  - usually learn to live w/ fictitious forces to retain an earth-centric view