Rotation of Rigid Bodies

PH301 F18
L32/ch10

Known results

$$\text{CM}: \quad \vec{R} = \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha \longrightarrow \frac{1}{M}\int \vec{r}\, dm$$

$$\Rightarrow \quad \vec{P} = M\dot{\vec{R}}, \quad \vec{F}^{ext} = M\ddot{\vec{R}}$$

prev HW: $\vec{L} = \vec{L}(\text{motion of CM}) + \vec{L}(\text{motion rel CM})$

- Consider planet around Sun. Sun ≈ fixed posn

$$\Rightarrow \quad \vec{L} = \vec{L}_{orbit} + \vec{L}_{spin}$$

$\dot{\vec{L}}_{orb} = \vec{R} \times \vec{F}^{ext}$ – L_{orb} evolves like planet were a point particle, mass conc @ CM

(reality: not perfectly spherical, sun's gravity not perfectly uniform, but excellent approx.)

$\dot{\vec{L}}_{spin} = \vec{\Gamma}^{ext}$ (about CM) net external torque is very small, so spin rate nearly constant

earth's equatorial bulge ⇒ small torque, leads to precession of rotation axis

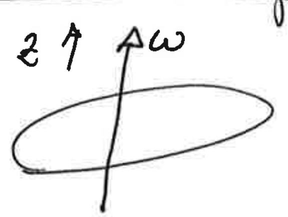
- Similar division in QM – spin & orbital L are often very nearly conserved separately

- $T = T(\text{motion of CM}) + T(\text{motion rel to CM})$

rigid body: rotation is the only possibility

- $U = U^{ext} + U^{int}$ $\quad U^{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta})$

Rotation about a fixed axis



divide body into tiny pieces m_α $(\alpha = 1 \cdots N)$

$\vec{L} = \sum_1' \vec{\ell}_\alpha = \sum_1' \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$

we know now $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$.

Let $\hat{z}$ be along $\vec{\omega}$, so $\vec{\omega} = (0,0,\omega)$ and $\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$

$\Rightarrow \vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$

so $\vec{\ell}_\alpha = m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = m_\alpha \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{vmatrix} = m_\alpha \omega(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2)$

now we can try to find $\vec{L}$. Start w/ L_z

$L_z = \sum_\alpha' m_\alpha \omega (x_\alpha^2 + y_\alpha^2)$ note $x^2 + y^2 = \rho^2 = (\text{dist from } z \text{ axis})^2$

ω indep of α

$L_z = \sum_1' m_\alpha \rho_\alpha^2 \omega \equiv I_z \omega$, $I_z = \sum_1' m_\alpha \rho_\alpha^2$ moment of inertia about z axis

$L = I\omega$ as before! and $I = \sum_1' (\text{mass})(\text{dist})^2$

now note: $T = \frac{1}{2} \sum_1' m_\alpha v_\alpha^2 = \frac{1}{2} \sum_1' m_\alpha (\omega^2 x_\alpha^2 + \omega^2 y_\alpha^2) = \frac{1}{2} \omega^2 \sum_1' m_\alpha \rho_\alpha^2$

or $T = \frac{1}{2} I_z \omega^2$ makes sense, rotation is about z and each m_α goes around a circle in the xy plane

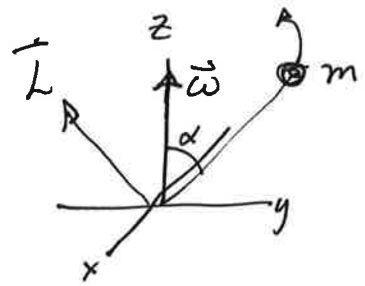
But... try it for x, y?

$L_x = \sum_\alpha' -m_\alpha \omega x_\alpha z_\alpha$ $\quad L_y = \sum_1' -m_\alpha \omega y_\alpha z_\alpha$ generally NOT zero!

What does this mean?

$\vec{\omega}$ points along $\hat{z}$ by construction, but $\vec{L}$ will point in a different direction if L_x and/or L_y are nonzero!

$\vec{L} = I\vec{\omega}$ is only true if I is not a scalar but a second rank tensor (3x3 matrix)!



- mass m on thin rigid rod
  $\vec{L} = m\vec{r}\times\vec{v}$ is not parallel to $\vec{\omega}$
  because $L_y \neq 0$ even though rotating about $\hat{z}$!
- since direction of $\vec{L}$ changes (precesses about $\vec{\omega}$) $\dot{\vec{L}} \neq 0$ and a torque is required just to keep the body rotating steadily!
- here $\dot{\vec{L}}$ is out of the page, CCW torque
- like a wheel rotating off center
- rotating with the body? m has centrifugal force ⟶ along $\hat{\rho}$
  ⇒ require CCW torque to keep rod holding m from breaking or bending – mass wants to fly off – not balanced
- like w/ 2D rotation – constraint on forces to maintain a specific path

Products of Inertia

- with rotation about z, clean up notation

$$L_x = I_{xz}\,\omega \qquad L_y = I_{yz}\,\omega$$

I_{ij} : i = axis for L, j = axis of rot

$$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha \qquad I_{yz} = -\sum m_\alpha y_\alpha z_\alpha$$

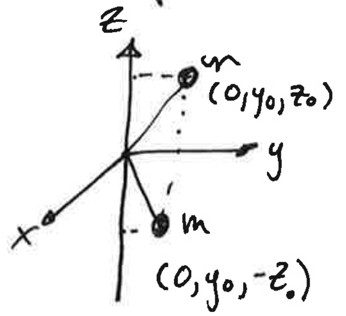
I_{xz}, I_{yz} = products of inertia

$I_{xz} \Rightarrow x$ component of $\vec{L}$ when rotation is about $\vec{z}$

Similarly $\quad I_{zz} = \sum' m_\alpha \rho_\alpha^2 = \sum' m_\alpha (x_\alpha^2 + y_\alpha^2)$

$$\Rightarrow \vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

Examples



- Single mass at $(0, y_0, z_0)$
- 2 equal masses at equal distances above xy plane rotating about $\vec{z}$
- Single mass:
  Since $x_\alpha = 0$ for all α, $I_{xz} = 0$

$$I_{yz} = -\sum' m_\alpha y_\alpha z_\alpha = -y_0 z_0 \sum' m_\alpha = -m y_0 z_0$$
$$I_{zz} = \sum' m_\alpha (x_\alpha^2 + y_\alpha^2) = y_0^2 \sum' m_\alpha = m y_0^2$$

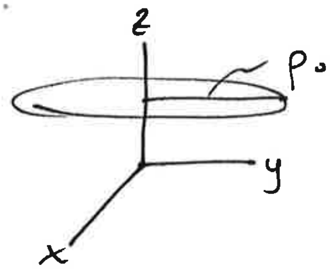
$\Rightarrow \vec{L}$ has nonzero y component

- 2 masses? $I_{xz} = 0$ still

$$I_{yz} = -\sum' m_\alpha y_\alpha z_\alpha = -m y_0 z_0 - m(-y_0) z_0 = 0$$
$$I_{zz} = 2 m y_0^2$$

no L_y component now - contributions of the 2 masses cancel "balanced"

- any body w/ reflection symmetry about $z=0$ plane will have $I_{xz} = I_{yz} = 0$ implied!
- if there is reflection symmetry about plane $\perp$ rot axis, then $\vec{L} \parallel \vec{\omega}$! i.e. symmetric objects rotate nicer no torque needed to keep it rotating
- in general, if an object has axial symmetry and rotates about this axis, then $\vec{L}$ and $\vec{\omega}$ are along that axis



ring of radius ρ_0

- reflection symmetry $\Rightarrow I_{xz} = I_{yz} = 0$!
  (each dm on one side cancels one on the opp. side)

$$I_{zz} = \sum' m_\alpha (x_\alpha^2 + y_\alpha^2) = \sum' m_\alpha \rho_\alpha^2 = \rho_0^2 \sum' m_\alpha = m \rho_0^2$$

Rotation about any axis

- Can always pick rotation axis to be z
  - problem? what if rotation axis changes w/ time?
  - Subtler: when $\vec{\omega}$ and $\vec{L}$ are in the same direction we say this is a principle axis, and choosing coord sys based on them makes life easier. BUT this takes away freedom to choose $\vec{\omega} \parallel \hat{z}$
  - so figure out how to let any axis be rot axis

let $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ e.g. spinning top w/ precession
thrown object spinning

$$\vec{L} = \sum' m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum' m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$ use "BAC-CAB" rule for triple prod.

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \Big[(y^2+z^2)\omega_x - xy\omega_y - xz\omega_z,\ -yx\omega_x + (z^2+y^2)\omega_y - yz\omega_z,$$
$$-zx\omega_x - zy\omega_y + (x^2+y^2)\omega_z\Big]$$

has form $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

: etc

$I_{xx} = \sum' m_\alpha (y_\alpha^2 + z_\alpha^2)$ I_{yy}, I_{zz} similar

$I_{xy} = -\sum' m_\alpha x_\alpha y_\alpha$ and so on

Simpler: $L_i = \sum_{j=1}^{3}{}' I_{ij}\,\omega_j$ or $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

moment of inertia tensor

implies $\vec{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}$, $\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ and $\vec{L} = \underline{\underline{I}}\,\vec{\omega}$

now look at off-diagonal terms

$$I_{xy} = -\sum{}' m_\alpha x_\alpha y_\alpha = -\sum{}' m_\alpha y_\alpha x_\alpha = I_{yx} \Rightarrow I_{ij} = I_{ji}$$

inertia tensor unchanged by reflection along main diagonal

$\Rightarrow$ I is its own transpose $\underline{\underline{I}} = \underline{\underline{\tilde{I}}}$

- that I is symmetric is key for analysis

$\Rightarrow$ we can find some axis that makes I diagonal, and along this <u>principle axis</u> $\vec{L} \parallel \vec{\omega}$

Cube rotating about a corner - mass M, side a. Sum $\to \int$

$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho(y^2+z^2) \quad \text{with } \rho = \frac{M}{a^3} = \text{mass density}$$

$$= \rho \iiint y^2\,dx\,dy\,dz + \rho \iiint z^2\,dx\,dy\,dz$$

$$= \rho\left(\frac{xzy^3}{3} + \frac{xyz^3}{3}\right)\Bigg|_0^a = \frac{2}{3}\rho a^5 = \frac{2}{3}Ma^2 \quad \text{same for } I_{yy}, I_{zz}$$

$$I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz\,(\rho x y) = -\rho\left(\tfrac{1}{2}x^2\right)\left(\tfrac{1}{2}y^2\right)(z)\Big|_0^a$$

$$= -\frac{1}{4}\rho a^5 = -\frac{1}{4}Ma^2 \quad \text{same for all off diagonal elements}$$

$$\Rightarrow I = Ma^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} \quad \text{about a corner.}$$

rotation about x? $\vec{\omega} = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{L} = \underline{\underline{I}}\,\vec{\omega} = Ma^2 \begin{bmatrix} \frac{2}{3}\omega \\ -\frac{1}{4}\omega \\ -\frac{1}{4}\omega \end{bmatrix}$

or $\vec{L} = Ma^2\omega\left(\frac{2}{3}, -\frac{1}{4}, -\frac{1}{4}\right)$ $\quad \vec{L}$ is not parallel to $\vec{\omega}$

along main diagonal? $\vec{\omega} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\omega$ $\quad \vec{L} = \frac{Ma^2}{6}\vec{\omega}$ they are parallel

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$\Rightarrow$ about center, $\vec{L} \parallel \vec{\omega}$ no matter what the direction of $\vec{\omega}$ is!

moving origin to center from edge diagonalizes $\underline{\underline{I}}$

Homework: about edge, unit vector is $(0,0,1)$, so $\vec{\omega} = (0,0,\omega)$

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega} \Rightarrow \vec{L} = Ma^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \left(-\frac{\omega}{4}, -\frac{\omega}{4}, \frac{2\omega}{3}\right) Ma^2$$

So for edge along z, $L_z = Ma^2 \cdot \frac{2\omega}{3}$, $I_{zz} = \frac{2}{3}Ma^2$