PH301 F18
L33 ch10

Cube rotated about a corner: we had

$$\underline{\underline{I}} = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

rotate about x? $\vec{L} = \underline{\underline{I}}\vec{\omega} = \frac{Ma^2\omega}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{Ma^2}{12}\begin{bmatrix} 8\omega \\ -3\omega \\ -3\omega \end{bmatrix}$

- so $\vec{L}$ is not parallel to $\vec{\omega}$
- I not diagonal $\Rightarrow$ nonzero L_y, L_z $\Rightarrow$ torque required to keep it rotating

rotate about main diagonal, $\hat{u} = \frac{1}{\sqrt{3}}(1,1,1)$? $\vec{\omega} = \omega\hat{u}$

$$\vec{L} = \frac{Ma^2}{12}\frac{\omega}{\sqrt{3}}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{Ma^2\omega}{12\sqrt{3}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$ now $\vec{\omega} \parallel \vec{L}$!

what if origin were at center of cube? $\vec{L}$ depends on choice of origin

$$I_{xx} = \frac{M}{a^3}\iiint_{-a/2}^{a/2}(r^2-x^2)\,dx\,dy\,dz = \frac{M}{a^3}\iint_{-a/2}^{a/2}(y^2+z^2)\,dy\,dy\underbrace{\int_{-a/2}^{a/2}dx}_{a}$$

$$= \frac{M}{a^2}\iint_{-a/2}^{a/2}(y^2+z^2)\,dy\,dy = \frac{M}{a^2}\cdot\frac{1}{3}\left(y^3+z^3\right)\Big|_{-a/2}^{a/2} = \frac{Ma^2}{6}$$

$$I_{xy} = \frac{M}{S^3}\iiint_{-a/2}^{a/2}(-xy)\,dx\,dy\,dy = 0$$ odd function

$$\Rightarrow \underline{\underline{I}} = \frac{Ma^2}{6}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{Ma^2}{6}\underline{\underline{1}}$$ or $\underline{\underline{I}}_{ij} = \frac{Ma^2}{6}\delta_{ij}$

- so $\vec{L}$ is parallel to $\vec{\omega}$ no matter what direction $\vec{\omega}$ has! result of high degree of symmetry of cube center
- always true if $\underline{\underline{I}}$ is diagonal

Principle axes of inertia

- when for an axis $\vec{L}$ is parallel to $\vec{\omega}$ $\Rightarrow$ principle axis
  i.e. $\vec{L} = \lambda \vec{\omega}$ for some real λ
- if $\vec{\omega}$ points along a principle axis, $\vec{L} = \lambda\vec{\omega}$ where
  λ = moment of inertia about that axis
- if inertia tensor is diagonal, (x, y, z) are principle axes
  and vice versa (e.g. cube about center)
- if a body has a symmetry axis through O (e.g. spinning top)
  $\Rightarrow$ that axis is a principle axis
  $\Rightarrow$ any 2 axes $\perp$ to it are too
- not only about symmetry: for any rigid body and any point O there are 3 perpendicular axes through O that make $\underline{\underline{I}}$ diagonal and $\vec{\omega} \parallel \vec{L}$
- results from $I_{ij} = I_{ji}$ - real symmetric matrix
  can always be made diagonal for the proper set of axes (find eigenvectors, diagonalize, ...)
- In these cases $I = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ λ_i = principle moments

  can show $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\tilde{\vec{\omega}}\,\underline{\underline{I}}\,\vec{\omega}$ in general
  and when I is diagonal, $T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$
- "diagonal when the thing balances"

- What are the principle axes? (eigenvectors of $\underline{\underline{I}}$; spoilers)

if $\vec{L}$ and $\vec{\omega}$ align, then $\vec{L} = \lambda\vec{\omega}$ for some real scalar λ

$\Rightarrow \underline{\underline{I}}\,\vec{\omega} = \lambda\vec{\omega}$ or (matrix)(vector) = (number)(same vector)

eigenvalue equation - operate on vector, get back (num)(vec)

- 2 parts: eigen<u>vectors</u> = principle axes
  eigen<u>values</u> = moments of inertia about principle axes

- to start: $\vec{\omega} = \underline{\underline{1}}\,\vec{\omega}$, so $\underline{\underline{I}}\,\vec{\omega} = \lambda\,\underline{\underline{1}}\,\vec{\omega}$ now we can subtract...

$\Rightarrow (\underline{\underline{I}} - \lambda\underline{\underline{1}})\vec{\omega} = 0$ nonzero solutions if $\det(\underline{\underline{I}} - \lambda\underline{\underline{1}}) = 0$

- characteristic ("secular") eqn for $\underline{\underline{I}}$ as we had with oscillations
- det is a cubic in λ since I is 3×3
  $\Rightarrow$ 3 solns, $\lambda_1, \lambda_2, \lambda_3$

$\boxed{\text{Example}}$ Cube about the corner. What are principle axes, etc.?

recall $\underline{\underline{I}} = \mu\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$ with $\mu = \frac{Ma^2}{12}$

$$\underline{\underline{I}} - \underline{\underline{1}}\lambda = \begin{bmatrix} 8\mu-\lambda & -3\mu & -3\mu \\ -3\mu & 8\mu-\lambda & -3\mu \\ -3\mu & -3\mu & 8\mu-\lambda \end{bmatrix} = \left.\begin{bmatrix} (11\mu-\lambda) & -(11\mu-\lambda) & 0 \\ 0 & (11\mu-\lambda) & -(11\mu-\lambda) \\ -3\mu & -3\mu & 8\mu-\lambda \end{bmatrix}\right\}\begin{matrix}\text{row } 1-2 \\ 2-3\end{matrix}$$

$$\det(\underline{\underline{I}} - 1\lambda) = (8\mu-\lambda)\left[(8\mu-\lambda)^2 - 9\mu^2\right] - (-3\mu)\left[(-3\mu)(8\mu-\lambda) - 9\mu^2\right]$$
$$+ (-3\mu)\left[9\mu^2 + 3\mu(8\mu-\lambda)\right]$$

yuck.

after row operations ... could do more yet w/column operation

$$\det = (11\mu-\lambda)\left[(11\mu-\lambda)(8\mu-\lambda) - 3\mu(11\mu-\lambda)\right] + (11\mu-\lambda)\left[(-3\mu)(11\mu-\lambda)\right]$$

$$= (11\mu-\lambda)^2(8\mu-\lambda-3\mu-3\mu) = (11\mu-\lambda)^2(2\mu-\lambda)$$

$\Rightarrow$ roots of $\det(\underline{\underline{I}} - \lambda \underline{\underline{1}}) = 0$, the eigenvalues are $\lambda_1 = 2\mu$, $\lambda_2 = \lambda_3 = 11\mu$

plug these eigenvalues in. for λ_1,

$$(\underline{\underline{I}} - \lambda \underline{\underline{1}})\vec{\omega} = \mu \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} 2\omega_x - \omega_y - \omega_z &= 0 \\ -\omega_x + 2\omega_y - \omega_z &= 0 \\ -\omega_x - \omega_y + 2\omega_z &= 0 \end{aligned}$$

subtract (first two), $\omega_x = \omega_y$ which implies $\omega_x = \omega_y = \omega_z$ from 1st eqn

$\Rightarrow$ principle axis is $(1,1,1)$ dir

i.e. body diagonal we knew

- primitive axis unit vector $\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)$

  $\lambda_1 = 2\mu = \frac{1}{6}Ma^2$ is moment of inertia rotating about main diagonal through center

- the other 2 eigenvalues are equal, $\lambda_2 = \lambda_3 = 11\mu$

$$(\underline{\underline{I}} - \lambda \underline{\underline{1}})\vec{\omega} = \mu \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

- all 3 eqns are $\omega_x + \omega_y + \omega_z = 0$
- does not uniquely specify direction of $\vec{\omega}$

- $\omega_x + \omega_y + \omega_z = \vec{\omega}\cdot\vec{e}_1 = 0$ just says $\vec{\omega}$ just has to be orthogonal to $\hat{e}_1$

$\Rightarrow$ any 2 orthogonal directions $\hat{e}_2, \hat{e}_3 \perp \hat{e}_1$ are fine for both, $\lambda_2 = \lambda_3 = 11\mu = \frac{11}{12}Ma^2$ = principle moments

- freedom of choice happens because $\lambda_2 = \lambda_3$ degeneracy if all 3 were different, all 3 directions unique

- what is the inertia tensor with the $\hat{e}_i$ as our coordinate axes?

$$I' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \frac{1}{12}Ma^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

- Once we know principle axes, no need to formally diagonalize since we know it has this form

- if all 3 eigenvalues are the same? any axis is principle true for cube or sphere about center

- for cube about center, $\underline{\underline{I}} = \frac{Ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \underline{\underline{I}} - \lambda\underline{\underline{1}} = \frac{Ma^2}{6}\begin{bmatrix} 1-\mu & 0 & 0 \\ 0 & 1-\mu & 0 \\ 0 & 0 & 1-\mu \end{bmatrix}$

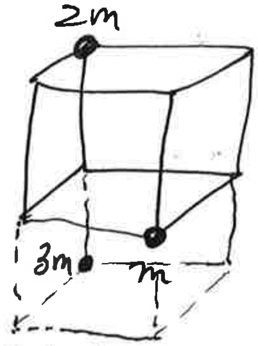
$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$ etc $\quad \det = (1-\mu)^3$ triply degenerate – free choice of any axis to start

Extra example

10.35 3 masses: m at $(a,0,0)$

$2m$ at $(0,a,a)$

$3m$ at $(0,a,-a)$



Clearly no reflection symmetry - axes not obvious

$$I_{xx} = \sum_i' m_\alpha (y_\alpha^2 + z_\alpha^2) = m(0) + 2m(a^2+a^2) + 3m(a^2+a^2) = 10ma^2$$

$$I_{yy} = \sum_i' m_\alpha (x_\alpha^2 + z_\alpha^2) = ma^2 + 2ma^2 + 3ma^2 = 6ma^2$$

$$I_{zz} = \sum_i' m_\alpha (y_\alpha^2 + x_\alpha^2) = 6ma^2$$

$$I_{yz} = -\sum_i' m_\alpha y_\alpha z_\alpha = 0 - 2ma^2 - 3m(-a^2) = ma^2 = I_{zy}$$

$$I_{zx} = -\sum_i' m_\alpha x_\alpha z_\alpha = I_{xy} = I_{xz} = I_{yx}$$

$$\Rightarrow \underline{\underline{I}} = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

presume eigenvalues are $\lambda = ma^2 \lambda'$ to avoid const.

$$\Rightarrow (\underline{\underline{I}} - \lambda \underline{\underline{1}}) = ma^2 \begin{bmatrix} 10-\lambda' & 0 & 0 \\ 0 & 6-\lambda' & 1 \\ 0 & 1 & 6-\lambda' \end{bmatrix} = (10-\lambda') \begin{vmatrix} 6-\lambda' & 1 \\ 1 & 6-\lambda' \end{vmatrix} - 0 \times \ldots + 0 \times \ldots$$

$$= (10-\lambda')\left((6-\lambda^2)^2 - 1\right) = (10-\lambda')\left[36 - 12\lambda + \lambda^2 - 1\right]$$

$$= (10-\lambda')\left[\lambda^2 - 12\lambda + 36\right] = (10-\lambda')(\lambda'-7)(\lambda'-5)$$

$$\Rightarrow \lambda_1 = 10ma^2, \quad \lambda_2 = 7ma^2, \quad \lambda_3 = 5ma^2$$

λ_1: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$

$$\left.\begin{array}{r} 0=0 \\ -4a_2 + a_3 = 0 \\ a_2 - 4a_3 = 0 \end{array}\right\} \Rightarrow a_2 = a_3 = 0$$

$\vec{a} = (a_1, 0, 0)$

$\Rightarrow \hat{e}_1 = (1,0,0)$

i.e. x axis, points at m

Similarly for λ_2

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$

$3a_1 = 0$

$-a_2 + a_3 = 0$

$\Rightarrow \frac{1}{\sqrt{2}}(0,1,-1) = \hat{e}_2$

$\frac{1}{\sqrt{2}}(0,-1,1) = \hat{e}_3$

2^{nd} axis points toward 2m, 3^{rd} toward 3m