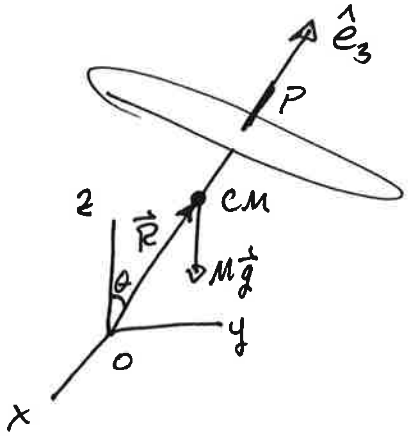
Precession of a top due to a weak torque



- top pivots freely at its tip O, makes angle θ wrt vertical
- top has axial symmetry, so $\underline{\underline{I}}$ is diagonal

$$\underline{\underline{I}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\hat{e}_3$ = symmetry axis
$\hat{e}_1, \hat{e}_2$ just need to be $\perp \hat{e}_3$

- without gravity: let it spin along symmetry axis so
  $\vec{\omega} = \omega \hat{e}_3 \Rightarrow \vec{L} = \lambda_3 \vec{\omega} = \lambda_3 \omega \hat{e}_3$
  since there is no net torque, $\vec{L}$ is constant
  $\Rightarrow$ spins indefinitely w/ same angular velocity
- now turn on gravity - leads to a torque!
  - $\vec{\Gamma} = \vec{R} \times M\vec{g}$, $|\Gamma| = MgR\sin\theta$ in magnitude
  - dir? $\vec{\Gamma} \perp \hat{z}$, $\vec{\Gamma} \perp \hat{e}_3$ axis of top
  - suppose Γ is small (low M/R)
  - angular momentum starts to change, $\dot{\vec{L}} = \vec{\Gamma}$
- Since $\dot{\vec{L}} \neq 0$, $\vec{\omega}$ changes and ω_2, ω_1 are no longer zero!
  if Γ small, can still expect $(\omega_1, \omega_2) \ll \omega_3$
  so $\vec{L} \simeq \lambda_3 \omega \hat{e}_3$ still approximately true (small corr.)
  i.e. mainly still spins about axis, w/ slow wobble

- in this approx, $\vec{\Gamma} \perp \vec{L}$ (since $\vec{\Gamma} \perp \hat{e}_3$)
  - $\Rightarrow \vec{L}$ changes in direction but not magnitude
- given $\vec{L} = \lambda_3 \omega \hat{e}_3$, it must mean $\hat{e}_3$ changes direction while $\omega \simeq$ constant
  - $\Rightarrow \dot{\vec{L}} = \lambda_3 \omega \dot{\hat{e}}_3 = \vec{R} \times M\vec{g}$
- we know $\vec{R} = R\hat{e}_3$ and $\vec{g} = -g\hat{z}$, so

$$\dot{\hat{e}}_3 = \frac{MgR}{\lambda_3 \omega} \hat{z} \times \hat{e}_3 = \vec{\Omega} \times \hat{e}_3 \quad \text{with } \vec{\Omega} = \frac{MgR}{\lambda_3 \omega} \hat{z}$$

this says the axis of the top $\hat{e}_3$ rotates w/ angular velocity $\vec{\Omega}$ about $\hat{z}$

- gravity's torque causes the top's axis to precess - it moves slowly around a vertical cone with fixed angle θ at angular frequency $\Omega = \frac{RMg}{\lambda_3 \omega}$
- grav torque is clockwise, $\vec{\Gamma}$ into board, and so is change of $\vec{L}$
- earth's equatorial bulge + spin axis inclination of $\theta = 23°$ causes earth's axis to precess - 1 turn in 26,000 years

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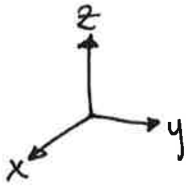
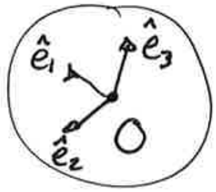
Euler's equations

- equations of motion for a rotating rigid body
- 2 key situations
  1) body pivoted about 1 end, like the top
  2) body w/o fixed point, like a flying baton, where we choose to look @ rotational motion about CM





(x,y,z) = space frame – inertial
(e_1, e_2, e_3) = noninertial body frame
- choose to be the principle axes!

- now body frame is rotating, but Ch 9 showed us how to handle in body frame $\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$
- if torque on body is $\vec{\Gamma}$, as seen in the space frame we need

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma}$$

from ch 9, we know

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \boxed{\dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}}$$

This is Euler's eqn. By components along princ. axes $\hat{e}_i$,

$$\left.\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned}\right\} \text{Euler's equations}$$

- Determine motion of a spinning body as seen from a frame fixed in the body
- Difficult to use – Γ_i are complex and unknown functions of time in the body frame, ω_i are coupled
- Easy if $\vec{\Gamma}=0$, or high degree of symmetry. E.g. top

$\vec{\Gamma}\perp\hat{e}_3 \Rightarrow \Gamma_3 = 0$. by symmetry $\lambda_1 = \lambda_2$

$\Rightarrow \lambda_3\dot{\omega}_3 = 0$ i.e. $\vec{\omega}$ component along symmetry axis is constant, as we assumed before!

Euler eqns with zero torque

Simpler form:

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$
$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$$

- first consider $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (all different), then $\lambda_1 = \lambda_2 \neq \lambda_3$
  Symmetry will simplify...
- Say all λ_i are different. At $t=0$, rotate about prin. axis $\vec{e}_3$
  $\Rightarrow \omega_1 = \omega_2 = 0$ at $t=0$
  $\Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$ – $\vec{\omega}$ doesn't change!
  further, $\vec{L} = \lambda_3 \vec{\omega}$ so $\vec{L}$ is constant as seen in any inertial frame
  $\Rightarrow$ if a body starts rotating about a principle axis, and has no torque, it continues to do so w/ const ω
  $\Rightarrow$ body can freely rotate about a principle axis at const. ω
- Suppose you nudge it a bit while rotating about princ. axis?
  $\Rightarrow$ small ω_1, ω_2

3rd eqn: $\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\underbrace{\omega_1\omega_2}_{\text{doubly small}} \simeq 0 \quad \Rightarrow \omega_3 \simeq \text{constant}$

1st two:

$$\left.\begin{aligned} \lambda_1 \dot{\omega}_1 &= [(\lambda_2 - \lambda_3)\omega_3]\,\omega_2 \\ \lambda_2 \dot{\omega}_2 &= [(\lambda_3 - \lambda_1)\omega_3]\,\omega_1 \end{aligned}\right\}\ \text{terms in [ ] approx. constant}$$

differentiate 1st eqn, then sub into second
– either SHM or EXP differential equation

$$\ddot{\omega}_1 = -\underbrace{\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]}\omega_1$$

(+)
- if positive $\omega_1 \Rightarrow$ SHM. ω_1 has small oscillations and goes repeatedly through $\emptyset$
- same for ω_2
- if body spins about princ axis w/ largest or smallest moment, motion is stable against small disturbances
  - just picks up a little wobble

($\lambda_3 > \lambda_1, \lambda_2$ or $\lambda_3 < \lambda_1, \lambda_2 \Rightarrow$ (+) in brackets)

(−) if λ_3 is in between λ_1, λ_2, $[\ ] < 0$
- solution for ω_1 is a real exponential (same for ω_2)
- moves rapidly away from $\emptyset$ (because e^{-t} soln violates b.c.s)
- spinning about intermediate principle axis is unstable!

- try it with a book - about intermediate axis, tumbles wildly

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<u>2 equal moments</u> (like the top) (all 3 different complex, doesn't add much)

let $\lambda_1 = \lambda_2 \Rightarrow \dot{\omega}_3 = 0$

- so ω_3 = constant about $\hat{e}_3$ body axis
- just 2 Euler eqns then

$$\dot{\omega}_1 = \frac{(\lambda_1-\lambda_3)\omega_3}{\lambda_1}\omega_2 = \Omega_b\omega_2$$

$$\dot{\omega}_2 = -\frac{(\lambda_1-\lambda_3)\omega_3}{\lambda_1}\omega_1 = -\Omega_b\omega_1$$

$$\Omega_b = \frac{\lambda_1-\lambda_3}{\lambda_1}\omega_3$$

Solve as usual w/ complex variable $\eta = \omega_1 + i\omega_2$

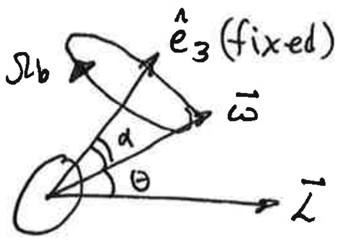
$\Rightarrow \quad \dot\eta = -i\Omega_b \eta$, and $\eta = \eta_0 e^{-i\Omega_b t}$

Set $\omega_1 = \omega_0$, $\omega_2 = 0$ at $t=0 \Rightarrow \eta_0 = \omega_0$

$$\Rightarrow \quad \vec\omega = (\underbrace{\omega_0 \cos\Omega_b t, \; -\omega_0 \sin\Omega_b t}_{\text{rotate w/ angular velocity } \Omega_b}, \; \underbrace{\omega_3}_{\text{constant}})$$

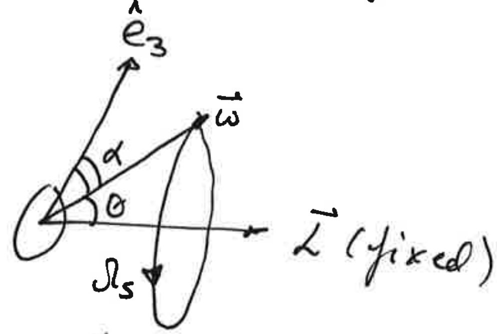
- Since ω_0, ω_3 are constant, the angle α between $\vec\omega$ and $\hat e_3$ is too

  $\Rightarrow$ $\vec\omega$ moves steadily about the <u>body cone</u> at Ω_b



body frame

$\vec\omega$ about $\hat e_3$



space frame

$\vec\omega$ about $\vec L$ $\quad \left(\Omega_s = \frac{L}{\lambda_1}\right)$

$$\vec L = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) = (\lambda_1\omega_0 \cos\Omega_b t, \; -\lambda_1\omega_0 \sin\Omega_b t, \; \lambda_3\omega_3)$$

- $\vec\omega$, $\vec L$, $\hat e_3$ lie in a single plane, angles between any 2 constant in time
- space frame, $\vec L$ is constant so $\vec\omega$ precesses about it
- free precession – not due to any torque
  due to unequal moments + $\vec\omega$ not aligned with symmetry axis