University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 3: Ch. 2.1-3

27 Aug 2018

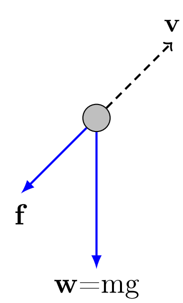
1 The Character of Air Resistance

We knew very well ignoring air resistance is silly much of the time.

A baseball launched at 25° at 110 mph should go 620 ft. This has likely never been done. All-time longest home runs are $15-20\%$ less than this, not to mention everyday launches with similar launch parameters are far shorter $(350-400\,\text{ft})$

What are the basic properties of the force of air resistance f?

- depends on speed, clearly
- opposite $\mathbf{v}$ for our purposes (not true in general - spinning objects, lift on airplane wings)
- depends on size or cross section/shape (hand out the car window | vs —)
- depends on surface finish (smooth or rough?)
- will depend on fluid properties - density and/or viscosity (only air for now)



Air resistance **f** always acts to oppose the velocity **v**.

Summarizing the figure above, $\mathbf{f} = -f(v)\,\hat{\mathbf{v}}$, where as before $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$. Essentially $\hat{\mathbf{v}}$ defines the direction of travel, and $\mathbf{f}$ is always in the opposite direction.

So what is **f**? All the previous dependencies ...complex! But, if well below the speed of sound in the medium, we have a good low-order approximation:

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}} \tag{1}$$

Of course it is obvious from a Taylor series approach this has to work if the velocity is "small enough."

f_{lin}: The first term is due to the viscous drag of the medium, it is proportional to the *viscosity* of the medium and the *size* of the particle.

f_{quad}: The second term is due to having to accelerate the fluid in front of you to get it out of the way - proportional to the *density* of the medium and the *cross-sectional area* of the particle. For a sphere:

$$\text{sphere} \qquad b = \beta D \tag{2}$$
$$c = \gamma D^2 \tag{3}$$
$$D = \text{diameter}; \quad \beta, \gamma \text{ depend on medium} \tag{4}$$
$$\text{air at STP} \qquad \beta = 1.6 \times 10^{-4}\,\text{N}\cdot\text{s/m}^2 \tag{5}$$
$$\gamma = 0.25\,\text{N}\cdot\text{s}^2/\text{m} \tag{6}$$

Often we can neglect one of the two terms depending on size/speed. Compare the magnitudes:

$$\frac{f_q}{f_l} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = \left(1.6 \times 10^3\,\text{s/m}^2\right) Dv \qquad \text{(air at STP only)} \tag{7}$$

Which one dominates depends on Dv - product of size and speed. Examples:

- **baseball**: $D = 7\,\text{cm}$, $v = 5\,\text{m/s}$ (slow): $f_q/f_l \sim 600$ can ignore linear term
- **oil drop**: $D = 1.5\,\mu\text{m}$, $v = 5 \times 10^{-5}\,\text{m/s}$, $f_q/f_l \sim 10^{-7}$ can ignore quadratic term
- **rain drop**: $D = 1\,\text{mm}$, $v = 0.6\,\text{m/s}$, $f_q/f_l \sim 1$ we are in trouble. need both.

Tiny & slow: linear dominates. slightly larger but more viscous? still linear. e.g., oil drop in air and ball bearing in molasses

most everyday projectiles? quadratic dominates.

By the way: $\frac{f_q}{f_l} \sim$ Reynold's number (same magnitude) - crossover from laminar to viscous flow.

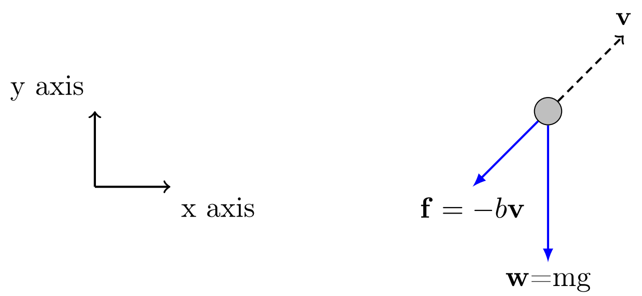
$$Re = \frac{Dv\rho}{\eta} = \left(1.6 \times 10^3\,\text{s/m}^2\right) \frac{f_q}{f_l} \frac{\rho}{\eta} \qquad \rho = \text{fluid density} \quad \eta = \text{kinematic viscosity} \tag{8}$$

f_{quad} dominates when Re is large (turbulent); f_{lin} dominates for small Re. We'll handle the linear regime first since it is a bit easier mathematically.

2 Linear Air Resistance

The force of air resistance is $\mathbf{f} = -b\mathbf{v}$. If we have only this and gravity, the force balance on a particle is simple enough

$$\sum \mathbf{F} = m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v} \tag{9}$$



Air resistance $\mathbf{f}$ always acts to oppose the velocity $\mathbf{v}$.

Neither force involves $\mathbf{r}$, so the equation of motion is independent of $\mathbf{r}$ (but NOT $\dot{\mathbf{r}}$ or $\ddot{\mathbf{r}}$). Noting $\ddot{\mathbf{r}} = \dot{\mathbf{v}}$,

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v} \tag{10}$$

This is a first order equation in $\mathbf{v}$. We can just solve it for $\mathbf{v}$ and integrate to get $\mathbf{r}$. With the $+y$ axis pointing upward, the component form is simple:

$$m\dot{v}_x = -bv_x \quad (A) \tag{11}$$
$$m\dot{v}_y = mg - bv_y \quad (B) \tag{12}$$

These are two uncoupled, linear, first-order differential equations. We can treat them separately, and they can be integrated. NOT so easy when $f \propto v^2$, then we have coupled equations.

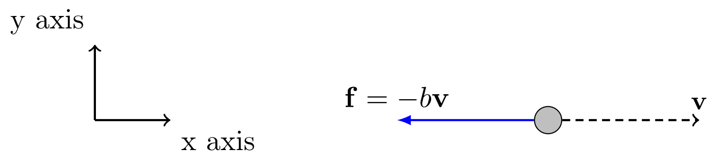
So we'll solve these for $v_x, v_y \to x(t), y(t) \to \mathbf{r}(t)$. But look again. The two component equations represent entire classes of problems on their own!

A : Any horizontal motion with drag but no gravity, e.g., a car

B : Any vertical motion with drag, e.g., dropped object

2.1 Case A: horizontal motion with drag

We assume the particle is by some means constrained to move on a horizontal surface so the y motion is irrelevant.



Air resistance **f** always acts to oppose the velocity **v**.

The equation is simple enough: $m\dot{v}_x = -bv_x$, which has the form $\frac{dv}{dt} = -kv$, with $k \equiv b/m$. We know this one . . . separate and integrate.

$$m\dot{v}_x = -bv_x = m\frac{dv_x}{dt} \tag{13}$$

$$\frac{dv_x}{v_x} = -k\,dt \qquad k \equiv \frac{b}{m} \tag{14}$$

$$\int \frac{dv_x}{v_x} = -\int k\,dt \tag{15}$$

$$\ln v_x = -kt + C \tag{16}$$

$$v_x(t) = Ae^{-kt} \qquad \text{rearrange and redefine constant, } A \equiv e^C \tag{17}$$

Since it was a first order equation, we expected one boundary condition - A. If we know the initial velocity $v(0) = v_{x0}$, then

$$v(0) = A \equiv v_{x0} \tag{18}$$

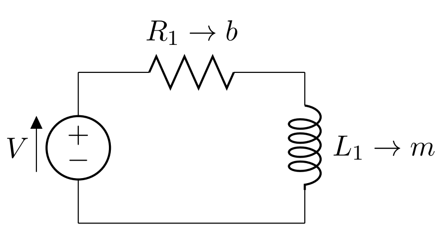
$$v_x(t) = v_{x0}e^{-kt} = v_{x0}e^{-t/\tau} \qquad \text{where} \qquad \tau = \frac{1}{k} = \frac{m}{b} \tag{19}$$

So the particle slows down exponentially with a characteristic decay time (or time constant) $\tau = m/b =$ (inertia)/(drag). We can also see that $v \to 0$ as $t \to \infty$, so the thing does stop eventually.

What we should also recognize is that this is the same equation we found for series RC or RL circuits in intro physics or engineering. If this were a circuit,

$$\text{series } RL \qquad R = b, L = m \qquad \tau = \frac{L}{R} \tag{20}$$

$$\text{series } RC \qquad R = \frac{1}{b}, C = m \qquad \tau = RC \tag{21}$$



The LC circuit is perhaps easier to understand: the electrical resistance and air resistance both serve as dissipative forces, while the inductor and mass serve as the inertia (mass). Check over your old course notes, it is the same first-order differential equation, just with different symbols.

We can use the same trick of separating variables and integrating once more to get $x(t)$. We know $v_x = \dot{x} = dx/dt$. If we choose our origin to be the particle's initial position, $x(0) = 0$, then

$$x(t) = x(0) + \int_0^t v_x(t')\,dt' \qquad \text{(we should distinguish integration variable and limit)} \tag{22}$$

$$x(t) = x(0) + \int_0^t v_{x0}\,e^{-t'/\tau}\,dt' = 0 + \left[-v_{x0}\tau e^{-t'/\tau}\right]_0^t \tag{23}$$

$$x(t) = v_{x0}\,\tau\left(1 - e^{-t/\tau}\right) \tag{24}$$

Now note that

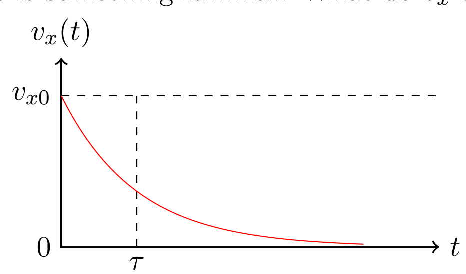
$$\lim_{t\to\infty} x(t) = v_{x0}\,\tau \equiv x_\infty \tag{25}$$

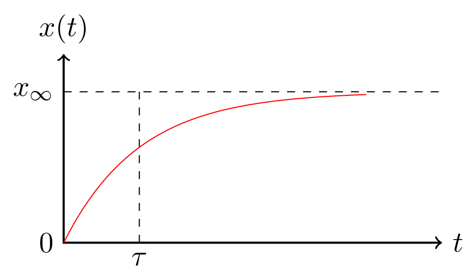
$$\implies \qquad x(t) = x_\infty\left[1 - e^{-t/\tau}\right] \tag{26}$$

Recall the voltage on the resistor in our RL circuit after switching on the voltage source:

$$V_R(t) = V\left[1 - e^{-t/\tau}\right] \qquad \tau = \frac{L}{R} \tag{27}$$

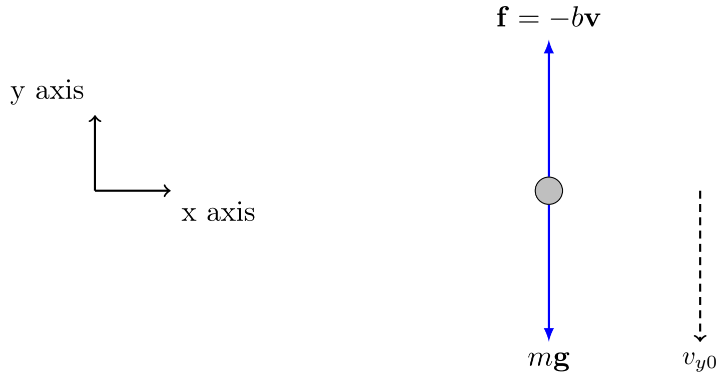
So this is something familiar. What do v_x and x look like as a function of time?





2.2 Case B: vertical motion

Let's consider an object thrown downward with velocity v_{y0}.



An object thrown downward with v_{y0}.

Now we have

$$m\dot{v}_y = mg - bv_y \tag{28}$$

Note we can have $m\dot{v}_y = m\ddot{y} = 0$ if $mg = bv_y$, so the drag and weight forces can balance! This is terminal velocity:

$$mg = bv_y \qquad \Longrightarrow \qquad \frac{mg}{b} = v_{y,\text{ter}} \equiv v_{\text{ter}} \tag{29}$$

For linear drag,

$$v_{\text{ter}} = \frac{\rho\pi D^2 g}{6\beta} \tag{30}$$

$$\text{oil drop: } D = 1.5\,\mu\text{m} \quad \Longrightarrow \quad v_{\text{ter}} \approx 10^{-4}\,\text{m/s} \tag{31}$$

$$\text{mist: } D = 0.2\,\text{mm} \quad \Longrightarrow \quad v_{\text{ter}} \approx 1\,\text{m/s} \tag{32}$$

$$\text{baseball: } D = 7.3\,\text{cm} \quad \Longrightarrow \quad v_{\text{ter}} \approx 95\,\text{mph} \quad \text{NOT linear drag} \tag{33}$$

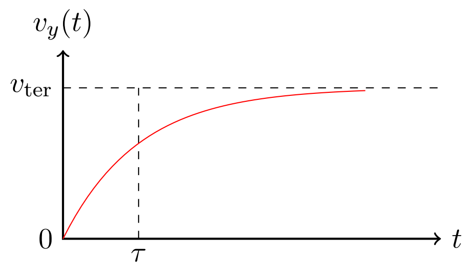
Solve it? Let $u = v_y - \frac{mg}{b}$, which means $\dot{u} = \dot{v}_y$. This gives

$$m\dot{u} = -b\dot{u} \tag{34}$$

This is the same equation as before, so the solution is the same. Do it again and back substitute, with $v_y(0) = v_{y0}$, $y(0) = 0$, and $v_{\text{ter}} = mg/b$. We have

$$v_y = v_{y0}e^{-t/\tau} + v_{\text{ter}}\left(1 - e^{-t/\tau}\right) = v_{\text{ter}} + \left(v_{y0} - v_{\text{ter}}\right)e^{-t/\tau} \tag{35}$$

From the first term we see that the initial velocity is also damped and that v_y approaches terminal velocity as $t \to \infty$. With an initial velocity of zero (dropped object), the velocity looks like this:

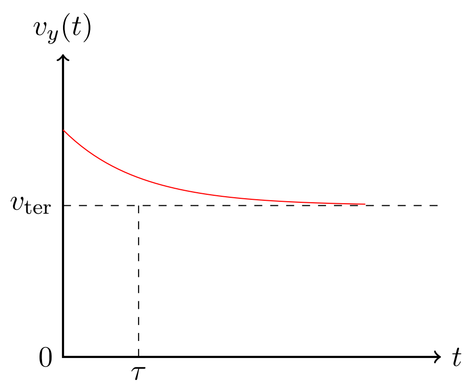


An object dropped with $v_{y0} = 0$.

Look again at $v_y(t)$, rearranging terms:

$$v_y = v_{y0} e^{-t/\tau} + v_{\text{ter}} \left(1 - e^{-t/\tau}\right) \tag{36}$$

The first term has the particle starting at v_{y0} but fading away exponentially. The second term starts at zero and reaches v_{ter} as $t \to \infty$. If launched downward with $v_y < v_{\text{ter}}$, the particle will speed up until it reaches v_{ter} and then continue at a constant velocity of v_{ter}. If launched downward with velocity greater than v_{ter}, the particle will slow down until reaching v_{ter} and thereafter continue with terminal velocity.



An object thrown downward with $v_{y0} > v_{\text{ter}}$.

Let's say we have a dropped object, $v_{y0} = 0$. How long does it take to reach terminal velocity, in terms of the time constant τ?

t	% v_{ter}
0	0
τ	63
2τ	89
3τ	95

By $3\tau = 3m/b$, within 95% of v_{ter}. For an oil drop, $\tau \sim 6\,\mu$s, so by $20\,\mu$s it is done. For mist, we are basically done in $20\,\mu$s. Note also since

$$v_{\text{ter}} = \frac{mg}{b} \qquad \tau = \frac{m}{b} \qquad \Longrightarrow \qquad v_{\text{ter}} = g\tau \tag{37}$$

So $v_{\text{ter}} = g\tau$ is the speed an object would acquire in time τ *if* the acceleration were just g without air resistance. The time constant τ is thus the characteristic time it would take to reach the same terminal velocity without air resistance. The smaller τ is, the stronger the influence of air resistance.

Integrate v_y once more,

$$y(t) = v_{\text{ter}}t + (v_{y0} - v_{\text{ter}})\,\tau\left(1 - e^{t/\tau}\right) \tag{38}$$

Combine this with $x(t)$ solution for purely horizontal motion and we have the motion of any projectile in a linear medium!

$$x(t) = v_{x0}\,\tau\left(1 - e^{-t/\tau}\right) \tag{39}$$

$$y(t) = (v_{y0} \pm v_{\text{ter}})\,\tau\left(1 - e^{t/\tau}\right) \pm v_{\text{ter}}t \qquad \pm = \text{launch up/down} \tag{40}$$

2.3 Trajectory and range in a linear medium

We have $x(t)$ and $y(t)$ as parametric equations, we can eliminate t to find $y(x)$:

$$y(x) = \left(\frac{v_{y0} + v_{\text{ter}}}{v_{x0}}\right)x + v_{\text{ter}}\tau\ln\left(1 - \frac{x}{v_{x0}t}\right) \tag{41}$$

Too complex to give much insight, but there it is. Here is a plot.

Figure 1: *Trajectory with linear air resistance compared to the trajectory in vacuum*

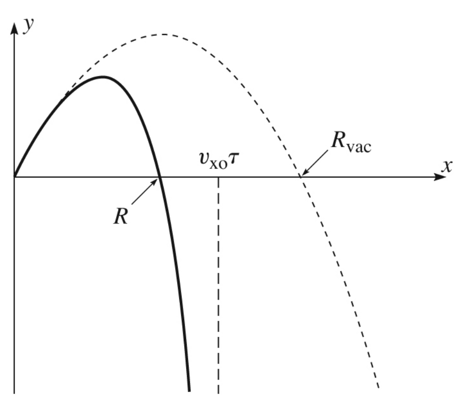


Figure 2.7
Taylor CLASSICAL MECHANICS

What is the range? it is x when $y = 0$, or

$$\left(\frac{v_{y0} + v_{\text{ter}}}{v_{x0}}\right) R + v_{\text{ter}}\tau \ln\left(1 - \frac{R}{v_{x0}t}\right) = 0 \tag{42}$$

This is a mess. The usual trick is to do a Taylor expansion, like $\ln(1-\epsilon) \approx -(\epsilon + \frac{1}{2}\epsilon^2 + \ldots)$, where $\epsilon = \frac{R}{v_{x0}t}$.

Neglect terms of $\mathcal{O}(\epsilon^3)$ and higher. Tedium ensues, and you can show

$$R = \frac{2v_{x0}v_{y0}}{g} - \frac{2}{3v_{x0}\tau}R^2 \approx \frac{2v_{x0}v_{y0}}{g} \tag{43}$$

The first term is the vacuum result we know. The second term is a reduction in range inversely proportional to τ, and so proportional to the air resistance term b. When τ is large or b is small, so when air resistance is negligible the range is basically what it is in vacuum.

We can simplify this though, for small air resistance:

$$R = R_{\text{vac}}\left(1 - \frac{4v_{y0}}{3v_{\text{ter}}}\right) \tag{44}$$

This makes it clear that air resistance reduces the overall range, of course. Since $v_{y0} \propto v$, the ratio of v/v_{ter} dictates the loss of range. If $v/v_{\text{ter}} \ll 1$ for the whole flight, air resistance is small. If $v/v_{\text{ter}} \gtrsim 1$ we almost certainly need air resistance included, and the approximation above for the range is no good!

2.4 Example

Consider a tiny metal pellet. Now it depends on the material, since $v_{\text{ter}} \propto \rho$:

$$D = 0.2\,\text{mm} \tag{45}$$

$$v = 1\,\text{m/s} \tag{46}$$

$$\theta = 45^\circ \tag{47}$$

$$\rho = 16\,\text{g/cm}^3 \quad \text{(gold)} \tag{48}$$

$$\rho = 2.7\,\text{g/cm}^3 \quad \text{(aluminum)} \tag{49}$$

In vacuum, for either:

$$R = \frac{2v_{x0}v_{y0}}{y} = \frac{2v_0^2 \sin\theta\cos\theta}{g} \approx 10.2\,\text{cm} \tag{50}$$

With air resistance:

$$\text{gold} \qquad \frac{4v_{y0}}{3v_{\text{ter}}} \sim \frac{4 \times 0.71}{3 \times 0.21} \sim 0.05 \qquad 5\% \text{ reduced to } 9.7\,\text{cm} \tag{51}$$

$$\text{aluminum} \qquad 30\% \text{ reduced to } \sim 7\,\text{cm} \tag{52}$$

3 HW problems