University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 4: Ch. 2.3-4

29 Aug 2018

1 Quadratic air resistance

Linear drag forces were OK for oil drops and small, slow things (when $Dv \lesssim 10^{-4}$). As you will see in HW1, quadratic drag forces are far more appropriate for most everyday situations. In this case,

$$\mathbf{f} = -cv^2\,\hat{\mathbf{v}} \tag{1}$$

$$m\dot{\mathbf{v}} = m\mathbf{g} + \mathbf{f} \tag{2}$$

How much difference could it make? Well, for $f = -cv^2$ we end up with a nonlinear differential equation, which is bad enough, but what is worse is that the x and y component equations are now *coupled* non-linear differential equations. They have no analytic solution in 2D. We will only be able to do special cases analytically (e.g., pure vertical or horizontal motion), but given a set of initial conditions, we can solve 2D motion with quadratic drag to arbitrary precision numerically. It isn't even that difficult, and we'll go through some numerical examples.

1.1 Horizontal motion with quadratic drag

We will neglect friction for now, and assume a particle is confined to a horizontal plane with only the force of air resistance. E.g., a bicycle coasting to a stop with frictionless wheels. This is a 1D problem, and we have

$$m\dot{v} = m\frac{dv}{dt} = -cv^2 \tag{3}$$

Though this is a nonlinear equation, we can still separate variables and integrate to solve for $v(t)$ and $x(t)$. In fact, this is generally true: if the overall force on a particle depends on velocity alone, $F = F(v)$, you can always separate variables to write the equation of motion as $m\,dv/F(v) = dt$ and integrate. Proceeding:

$$m\frac{dv}{v^2} = -c\,dt \tag{4}$$

$$m\int_{v_o}^{v} \frac{dv'}{(v')^2} = -\int_0^t dt' \tag{5}$$

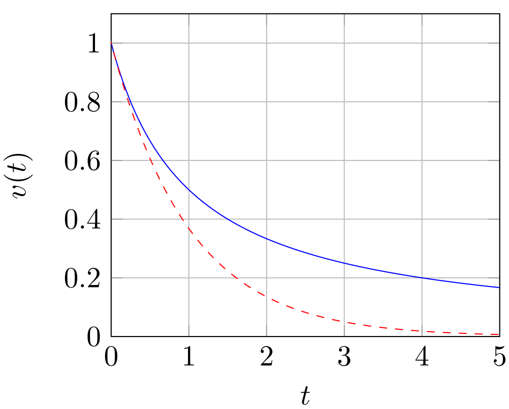
Here we can make the integral definite by assuming that the particle starts at $t = 0$ with velocity v_o, and we are interested in a later time t where the velocity is v. We also make the integration variables primed to keep them separate from the limits of the integral, which is perhaps pedantic but avoids confusion. The integral is simple enough:

$$m\left(\frac{1}{v_o} - \frac{1}{v}\right) = -ct \tag{6}$$

$$v(t) = \frac{v_o}{1 + cv_o t/m} = \frac{v_o}{1 + t/\tau} \tag{7}$$

$$\tau \equiv \frac{m}{cv_o} \tag{8}$$

We can see then that the velocity decays with time, and when $t = \tau$, $v = v_o/2$. As before, τ is an indicator of how long it takes for air resistance to slow the particle significantly. Here's a plot, with the blue curve being $v(t)$ as above, and the red curve being our previous result for linear air resistance. Without air resistance, the velocity would just be constant. You can easily see that linear air resistance slows the particle down much more effectively.

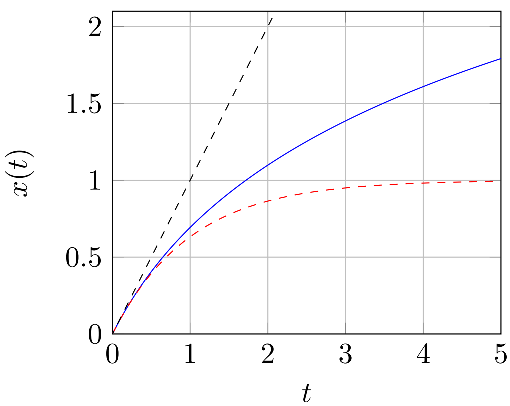


To get $x(t)$, we just integrate once more. We'll let $x(0) = 0$ for convenience.

$$x(t) = x(0) + \int_0^t v(t')\,dt' = \int_0^t \frac{v_o}{1 + t'/\tau}\,dt' = v_o\tau \ln\left(1 + \frac{t}{\tau}\right) \tag{9}$$

And once more a plot, comparing $x(t)$ for linear (red) and quadratic (blue) drag (with $x_\infty = v_o =$

$\tau = 1$) as well as the case with no air resistance (black):



Superficially the two cases are again similar, but we should look more carefully.

- both have $v \to 0$ as $t \to \infty$
- however: linear drag has v decreasing exponentially, while quadratic drag has v decreasing more slowly as $1/t$
- for $x(t)$, the linear case has the particle approaching a stopping point at x_∞, but the quadratic case has $x \to \infty$ as $t \to \infty$!
- we conclude that the particle will stop for linear drag, but will *not* stop for quadratic drag!

Why is this? If $f \propto v^2$, when v is small f is *very* small. That means at some point in time, v will be small enough that linear drag takes over and dominates thereafter. Recall

$$\frac{f_{\text{quad}}}{f_{\text{linear}}} \propto Dv \tag{10}$$

For large Dv, we are safely in the quadratic regime. As the particle slows, at some point we must reach a point where the linear term takes over. Once that happens, we do have a limiting position and the particle will stop. (Also: keep in mind we have ignored friction, which would eventually cause the particle to stop anyway.)

1.2 Vertical motion with quadratic drag

Let the downward direction be $+y$. We'll consider a particle moving downward at velocity v under the influence of quadratic drag and gravity. The equation of motion is then

$$m\dot{v} = m\frac{dv}{dt} = mg - cv^2 \tag{11}$$

Since we have two opposing forces, at some point v will be large enough that $mg = cv^2$ and we will have zero acceleration, $\dot{v} = 0$. This point is when we have reached terminal velocity, and from the previous equality, evidently

$$mg = cv_{\text{ter}}^2 \tag{12}$$

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}} \tag{13}$$

Using this, we can simplify the equation of motion. First divide through by m.

$$\dot{v} = \frac{dv}{dt} = g - \frac{c}{m}v^2 = g - \frac{c}{mg}gv^2 = g - \frac{v^2}{v_{\text{ter}}^2}g = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right) \tag{14}$$

Now we can use separation of variables again to find $v(t)$. Assume the ball starts from rest, $v(0) = 0$ for simplicity.

$$g\,dt = \frac{dv}{1 - v^2/v_{\text{ter}}^2} \tag{15}$$

$$\int_0^t g\,dt = gt = \int_0^v \frac{dv}{1 - v^2/v_{\text{ter}}^2} = \int_0^{uv_{\text{ter}}} \frac{v_{\text{ter}}\,du}{1-u^2} \qquad \text{let } u = v_{\text{ter}} \text{ so } du = dv/v_{\text{ter}} \tag{16}$$

This is a known integral you can look up. We'll do it anyway. There are two options: partial fractions, and trig substitution.

$$gt = \int_0^{uv_{\text{ter}}} \frac{v_{\text{ter}}\,du}{1-u^2} = v_{\text{ter}} \int_0^{uv_{\text{ter}}} \frac{du}{(1+u)(1-u)} = v_{\text{ter}} \int_0^{uv_{\text{ter}}} \frac{du}{2(1+u)} + \frac{du}{2(1-u)} \tag{17}$$

$$gt = v_{\text{ter}}\left[\frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u)\right] \tag{18}$$

It is a bit simpler if we recognize that the term in [] is the definition of $\operatorname{arctanh} u$.

$$gt = v_{\text{ter}} \operatorname{arctanh}(u) = v_{\text{ter}} \operatorname{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) \tag{19}$$

$$\implies \quad v(t) = v_{\text{ter}} \tanh\left(\frac{gt}{\text{ter}}\right) \tag{20}$$

Alternatively, you can go back to Eq. 16 and let $u = \tanh x$ such that $du = \operatorname{sech}^2(x)\,dx$:

$$gt = \int \frac{v_{\text{ter}}\, du}{1-u^2} = v_{\text{ter}} \int \frac{\text{sech}^2(x)\, dx}{1-\tanh^2(x)} = v_{\text{ter}} \int \frac{\text{sech}^2(x)\, dx}{\text{sech}^2(x)} = v_{\text{ter}} x \tag{21}$$

$$gt = v_{\text{ter}} \,\text{arctanh}(u) = v_{\text{ter}}\,\text{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) \tag{22}$$

Shorter, but the odds of remembering to do a hyperbolic tangent substitution are about as good as remembering how to do partial fractions, so normally speaking we would just look up the integral. We can integrate once more to find $y(t)$, which is actually simpler. Let $y(0) = 0$ for convenience.

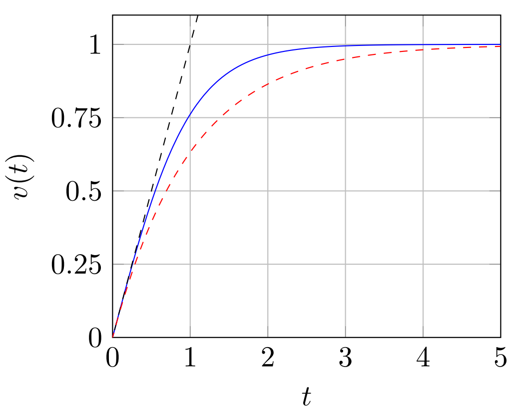
$$y(t) = \int_0^t v(t')\, dt' = v_{\text{ter}} \int_0^t \tanh\left(\frac{gt}{v_{\text{ter}}}\right) dt' = v_{\text{ter}} \int_0^t \frac{\sinh\left(\frac{gt}{v_{\text{ter}}}\right)}{\cosh\left(\frac{gt}{v_{\text{ter}}}\right)} dt' \quad \text{form } du/u \tag{23}$$

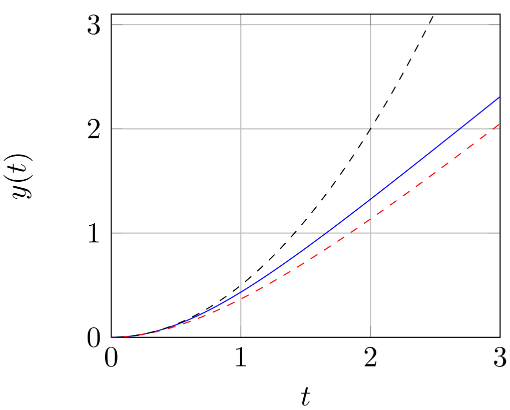
$$y(t) = \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right] \tag{24}$$

A beautiful result it is not. You can work out a few things though.

- $y(t)$ goes as $y \sim \frac{1}{2}gt^2$ for small t
- $y(t)$ approaches $y = v_{\text{ter}} + c$ for large t

We can also make a plot of $v(t)$ and $y(t)$, again comparing linear drag (red) and quadratic drag (blue) as well as the case where there is no air resistance (black; in all cases $g = v_{\text{ter}} = 1$). Once again, qualitatively similar.





For our usual example of a baseball,

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}} \qquad c = \gamma D^2 \qquad \gamma = 0.25\,\text{Ns}^2/\text{m}^4 \qquad \Longrightarrow \qquad v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} \tag{25}$$

$$D = 7\,\text{cm} \qquad m = 0.15\,\text{kg} \qquad \Longrightarrow \qquad v_{\text{ter}} \approx 35\,\text{m/s} \approx 78\,\text{mph} \tag{26}$$

Basically any MLB pitcher can throw faster than this, meaning that in all realistic cases the drag force on the baseball exceeds its weight. That means if you can control a fraction of the drag force by putting spin on the ball (owing to the fact that the laces make its surface uneven), you can make the baseball's trajectory do amazingly weird things. You can also work out that a dropped baseball reaches nearly terminal velocity within about 5 sec. By comparison, a pitched baseball reaches home plate in less than half a second. You can also work out that a dropped baseball loses about 50 m of distance compared to what you'd expect in vacuum in the first 6 sec (130 m vs 180 m in vacuum).

2 Trajectory with quadratic drag

With both horizontal and vertical motion, we can write down the equation of motion as well as the component x and y equations:

$$m\ddot{\mathbf{r}} = m\mathbf{g} - cv^2\,\hat{\mathbf{v}} = m\mathbf{g} - cv\mathbf{v} \tag{27}$$

$$m\dot{v}_x = -cv^2\cos\theta = -cvv_x = -c\sqrt{v_x^2+v_y^2}v_x = -cvv_x \tag{28}$$

$$m\dot{v}_y = -mg - c\sqrt{v_x^2+v_y^2}v_y = -cvv_y \tag{29}$$

Now we are in trouble: the x and y equations are coupled. Worse, neither is like the purely horizontal or vertical special cases we had before, so we can't just combine our previous solutions like we did for linear air resistance. In fact, there is no analytic solution at all! We *can* solve these equations numerically to arbitrary accuracy given initial conditions, however, there is just no general solution. Below is some python code to do the job we will go over in the next class, but I can give you the basic idea.

We start with some initial conditions, e.g., a starting value of $\mathbf{v}$ and position (usually the origin). We then take one tiny step forward in time dt and ask what the acceleration is. Over such a small time step, the change in velocity is not so much, so we can assume that in a time dt the velocity changes by $\Delta v_x = ax\,dt$ and $\Delta v_y = ay\,dt$. We add this correction to the velocity, and then do the same to find Δx and Δy. Then we add one more dt to the clock, and do it all again. Repeat until the particle is back at the ground. Here is some pseudocode to give you the idea.

define g;
define initial velocity v define $t = 0$;
define time step $dt \sim 10^{-3}\,\mathrm{s}$;
define drag coefficient, diameter, mass, etc.;
set initial position as zero (tossed from ground);
while *position > 0* **do**
    set acceleration, e.g., $a_x = -cvv_x$;
    update velocity, e.g., $v_{\mathrm{now}} = v_{\mathrm{prev}} + a\,dt$;
    update position, e.g., $x_{\mathrm{now}} = x_{\mathrm{prev}} + v_{\mathrm{prev}}t + \frac{1}{2}at^2$;
    add dt to time elapsed, e.g., $t_{\mathrm{now}} = t_{\mathrm{prev}} + dt$;
    store (t, x, y) to an array to build up the trajectory;
end
particle is now back on the ground.;
use (t, x, y) array to plot trajectory;

Based on that idea, what follows is some Python code that calculates a trajectory with and without air resistance and plots the result.

```
import math
import numpy as np
import matplotlib.pyplot as plt


def trajectory(m,c,vo,thetao):
        vx = vo*np.cos(thetao)
        vy = vo*np.sin(thetao)
        dt = 1e-5

        xtemp=0
        ytemp=0
        t=0
        yo=0
        y=[]
        x=[]

        while ytemp>=0:
                v = math.sqrt(vx**2+vy**2)
                theta = math.atan(vy/vx)                # current velocity and angle
                ax = -c*vx*v/m                          # update accel from current vel
                ay = -g-c*vy*v/m                        # ma = D(v dot v), quadratic drag
                vx += ax*dt                             # update velocity and position components
                vy += ay*dt
                xtemp += vx*dt+0.5*ax*dt*dt
                ytemp += vy*dt+0.5*ay*dt*dt
                if y<0:
                        y=0
                t+=dt
                x.append(xtemp)                         # store coord in arrays to generate path
                y.append(ytemp)
        return (x,y,xtemp)

T = 50                          #launch angle in degrees
g = 9.81                        #m/s^2
vo= 30                          #m/s
diam = 0.07                     #m
c = 0.25*diam**2                #drag coeff c = (gamma)D^2 = (0.25)(7cm)^2
m = 0.15                        #kg

#calculate trajectory for baseball with air resistance

x,y,range =trajectory(m,c,vo,np.radians(T))

print "Angle"
print "%.2f" % (T)
print "Range"
print  "%.2f" % (range)

plt.plot(x,y,label='with drag')

#run it again, with no air resistance for comparison, i.e., c=0

x,y,range =trajectory(m,0,vo,np.radians(T))  #vary launch angle

print "Angle"
print "%.2f" % (T)
```

```
print "Range"
print  "%.2f" % (range)

plt.plot(x,y,label='without drag',linestyle='dashed')

ymax = 1.25*vo**2/(4*g) #max height at 45deg launch with no drag
        #with 1.25 fudge factor, used to scale plot vertically

plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.ylim([0,ymax])
plt.legend(frameon=0)

plt.savefig('drag-ex2-6.pdf',bbox_inches='tight') #write to file
plt.show()  #write to screen
```