University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 4: Ch. 2.5-7

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1 Charge in a uniform B field

Another interesting case we can handle is a charge in a uniform **B** field, where the force is a function of velocity alone. We know the magnetic force law:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1}$$

Assume that **B** is uniform along the $\hat{\mathbf{z}}$ axis, so $\mathbf{B} = (0, 0, B)$, and we have a positive charge q with velocity v. The equation of motion is then

$$m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B} \tag{2}$$

$$\mathbf{v} = (v_x, v_y, v_z) \tag{3}$$

$$\mathbf{B} = (0, 0, B) \tag{4}$$

We can readily compute $\mathbf{v} \times \mathbf{B}$:

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = (v_y B, -v_x B, 0) \tag{5}$$

Then the equation of motion gives us three component equations:

$$m\dot{v}_x = qBv_y \tag{6}$$

$$m\dot{v}_y = -qBv_x \tag{7}$$

$$m\dot{v}_z = 0 \qquad \Longrightarrow \qquad v_z = v_{zo} = \text{const} \tag{8}$$

The z equation makes sense: since **B** only alters components perpendicular to it, along z we should have motion with constant velocity. It is also clear that we have separate equations for the components parallel to **B** (the z component) and the components perpendicular to **B** (those in the $x - y$ plane). Now define a parameter $\omega = qB/m$ and re-examine the x and y equations.

$$\dot{v}_x = \omega v_y \tag{9}$$

$$\dot{v}_y = -\omega v_x \tag{10}$$

The book solves these equations using complex numbers, which is very handy and you should definitely follow the book's solution. However, for illustrative purposes we'll solve these another way (which is actually what problem 2.54 asks you to do). First, take the time derivative of the second equation.

$$\frac{d\dot{v}_y}{dt} = \ddot{v}_y = -\frac{d(\omega v_x)}{dt} = -\omega \dot{v}_x \tag{11}$$

But we know from the first equation that $\dot{v}_x = \omega v_y$, so we have

$$\ddot{v}_y = -\omega \dot{v}_x = -\omega^2 \dot{v}_y^2 \tag{12}$$

This is the equation for simple harmonic motion, so we immediately know v_y oscillates sinusoidally:

$$v_y(t) = v_p \cos(\omega t + \delta) \tag{13}$$

Thus v_y oscillates sinusoidally with frequency ω and phase δ (with δ determined from boundary conditions) with amplitude v_p which is yet to be determined. Now, we should remember v_x is coupled to v_y via Eq. 10, which we can use to our advantage:

$$v_x(t) = -\frac{1}{\omega}\dot{v}_y = -\frac{1}{\omega}\frac{d}{dt}\left[v_p \cos(\omega t + \delta)\right] = v_p \sin(\omega t + \delta) \tag{14}$$

The equations for v_x and v_y together are the equation of a circle - the velocity vector in the $x - y$ plane executes uniform circular motion with angular velocity ω, and the velocity along z is constant as you recall. Further, we can now figure out what v_p is:

$$v_x^2 + v_y^2 = v_p^2 = v_\perp^2 \tag{15}$$

The amplitude v_p is just the velocity of the particle in the $x - y$ plane, i.e., the velocity component perpendicular to $\mathbf{B}$. We can easily integrate both to get position versus time, which will clearly also give uniform circular motion. Since we also know $v_z(t) = v_{zo}$, we can easily find z as well.

$$x(t) = x_o + \frac{v_\perp}{\omega}\cos(\omega t + \delta) \tag{16}$$

$$y(t) = y_o + \frac{v_\perp}{\omega}\sin(\omega t + \delta) \tag{17}$$

$$z(t) = z_o + v_{zo} t \tag{18}$$

This x and y equations describe a circle with center at $\mathbf{r}_o = (x_o, y_o)$ and radius

$$r = \frac{v_\perp}{\omega} = \frac{v_\perp}{qB/m} = \frac{m v_\perp}{qB} = \frac{p_\perp}{qB} \tag{19}$$

Which you should remember as the same result from introductory physics as the uniform circular motion of a particle confined to the $x - y$ plane with a constant $\mathbf{B} = B\,\hat{\mathbf{z}}$. Adding in the fact that $z(t)$ has linearly increasing velocity, the curve in three dimensions is a *helix* (or spiral) of pitch $2\pi v_{zo}$ and radius $r = mv_\perp/qB$. Positive charges spiral around $\mathbf{B}$ counterclockwise, while electrons spiral around $\mathbf{B}$ clockwise. Below is a picture, with the x and y simple harmonic motion components in red/green, and the resulting helix in blue.

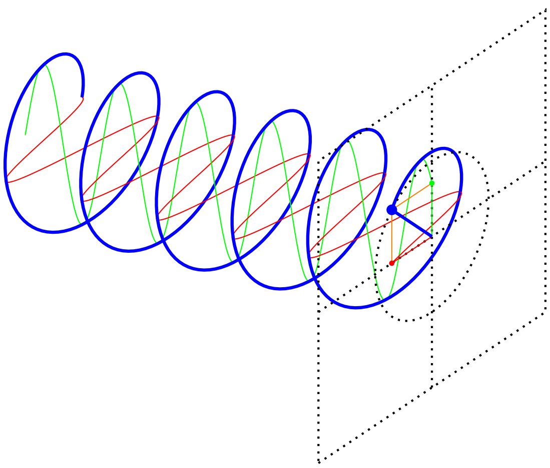


Figure 1: *Helical path of a charge in a constant* **B** *field. Code for this drawing by StackExchange user Mark Wibrow,* *https://tex.stackexchange.com/questions/99369/how-to-plot-circular-polarized-electromagnetic-wave.*

2 Example problems

1. Taylor 2.6 For a dropped object experiencing linear air resistance, we know $v_y(t) = v_{\text{ter}}(1 - \exp(-t/\tau))$. Show that for small v_y we recover the well-know vacuum result. Show the same for the position.

Solution: For a dropped object, small v corresponds to small t. For small t, we can approximate the exponential term by the first few terms of its Taylor series:

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \ldots \tag{20}$$

Using this result, with $x = -t/\tau$ and "small t" meaning $t \ll \tau$ so $x \ll 1$,

$$v_y \approx v_{\text{ter}}\left[1 - \left(1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2}\right)\right] = v_{\text{ter}}\left(\frac{t}{\tau} - \frac{t^2}{2\tau^2}\right) \qquad \text{note } \tau = v_{\text{ter}}/g \tag{21}$$

$$v_y \approx v_{\text{ter}}\left(\frac{tg}{v_{\text{ter}}} - \frac{t^2g^2}{2v_t^2}\right) = gt - \frac{g^2t^2}{2v_{\text{ter}}} \tag{22}$$

For small $t \ll \tau$, we can neglect the second term and we have $v_y = gt$. The second term represents the lowest-order correction to the velocity due to air resistance, and as we should expect it is negative.

For the position, we can just integrate the two terms we found for $v_y(t)$ - there is no real need to go back to $y(t)$ and approximate again, since we will end up approximating the same exponential functions. The integral of Eq. 22 is trivial, and

$$y(t) \approx \frac{1}{2}gt^2 - \frac{g^2t^3}{6v_{\text{ter}}} \tag{23}$$

Again, for $t \ll \tau$ we can neglect the second term, and we recover the usual result. The second term is again the lowest-order correction due to air resistance.

2. Taylor 2.7 If F is a function of v alone ($F = F(v)$), then you should be able to show

$$t = m\int\limits_{v_o}^{v} \frac{dv'}{F(v')} \tag{24}$$

Solution: We just need to write down the equation of motion, separate variables, and integrate.

$$m\dot{v} = F(v) = m\frac{dv}{dt} \tag{25}$$

$$dt = m\frac{dv}{F(v)} \tag{26}$$

$$\int\limits_{0}^{t} dt' = \int\limits_{v_o}^{v} m\frac{dv'}{F(v')} \tag{27}$$

$$t = m\int\limits_{v_o}^{v} \frac{dv'}{F(v')} \tag{28}$$

In the case $F = F_o$, i.e., a constant force, we find

$$t = \int\limits_{v_o}^{v} m\frac{dv'}{F_o} = \frac{m(v-v_o)}{F_o} \qquad \Longrightarrow \qquad F_o t = m(v-v_o) = \Delta p \tag{29}$$

This is just conservation of momentum - the impulse given by the force ($F_o t$) must equal the change in the particle's momentum. **3.** Taylor 2.39 **(a)** What if you have quadratic drag and friction?

Let $m = 80\,\text{kg}$, $c = 0.20\,\text{N}/\left(\text{m/s}\right)^2$, with a constant force of friction of $3\,\text{N}$. **(b)** If the particle starts with $v_o = 20\,\text{m/s}$, how long does it to slow down to 15, 10, and $5\,\text{m/s}$?

Solution: Take the force of friction to be constant, $f_{\text{fr}} = b$, and the quadratic drag force to be $f_q = -cv^2$. We write down the equation of motion, separate variables, and integrate as usual.

$$m\dot{v} = -cv^2 - b = m\frac{dv}{dt} \tag{30}$$

$$-dt = m\frac{dv}{cv^2 + b} \tag{31}$$

$$-t = m\int_{v_o}^{v} \frac{dv}{b/m + (c/m)v^2} = \sqrt{\frac{m^2}{bc}} \arctan\left(\sqrt{\frac{mc}{mb}}v\right)\Bigg|_{v_o}^{v} \tag{32}$$

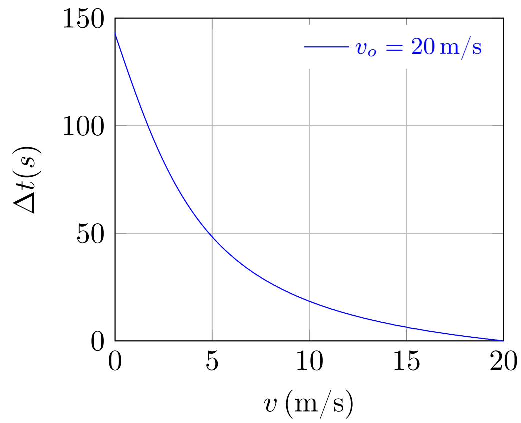
$$t = \frac{m}{\sqrt{bc}}\left[\arctan\left(v_o\sqrt{\frac{c}{b}}\right) - \arctan\left(v\sqrt{\frac{c}{b}}\right)\right] = \frac{m}{\sqrt{bc}} \arctan\left(\sqrt{\frac{b}{c}}\left[\frac{v - v_o)}{1 + cvv_o/b}\right]\right) \tag{33}$$

The last expression relies on an identity for $\arctan a - \arctan b$, viz.[i]

$$\arctan a - \arctan b = \arctan\left(\frac{a-b}{1+ab}\right) \tag{34}$$

Given $v_o = 20\,\mathrm{m/s}$ and the other parameters given, we can find the time it takes to slow down to 15, 10, and 5 m/s easily. May as well find the time to stop as well, and a plot for good measure.

v	Δt
15 m/s	6.34 s
10 m/s	18.4 s
5 m/s	48.3 s
0 m/s	143 s



3 Go over homework problems

[i] https://proofwiki.org/wiki/Difference_of_Arctangents