University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 7: Ch. 3.4-5

7 Sept 2018

1 Center of Mass

Recall from last time the definition of the center of mass:

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm = \frac{1}{M}\int \rho\mathbf{r}\,dV \tag{1}$$

where dV is a volume element. In principle the density could be a non-uniform throughout the object, such that $\rho = \rho x, y, z$, but ordinarily we will work with objects of uniform density. That it is a vector equation means you would typically do a separate integral for the x, y, and z axes, for example.

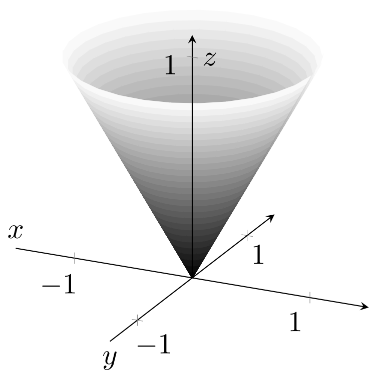
The center of mass can be thought of as the *first moment* of the matter distribution, with the net mass M being the zeroth order moment. We can make an analogy with the moments of statistical distributions: higher order moment describe the distribution with increasing specificity. The moment of order n goes as $\int r^n\,dm$, so higher order moments weight bits of mass farther from the origin more strongly. Whether the moment is odd or even then also tells us something about the symmetry of the object.

Moment	Statistical quantity	Physical quantity
0	overall probability = 1	mass
1	mean	center of mass
2	variance	moment of inertia
3	skewness	?
4	kurtosis (sharpness)	?

The third and fourth moments of the matter distribution have no standard names as far as I can tell. This is just an aside, so you can see that mass, center of mass, and moment of inertia are all closely related.

1.1 Center of mass of a cone

Consider a cone of uniform density with its apex at the origin oriented along the z axis, as shown below. The radius at its maximum height h is R, and so the radius at a lower height z will be $r = R(\frac{z}{h})$. By symmetry, we can say that $\overline{x} = \overline{y} = 0$, i.e., it is clear that the center of mass lies at $x = y = 0$ and along the z axis. That means we only need to do the integral for the z axis.



$$Z = \frac{1}{M}\int \rho z\, V = \frac{\rho}{M}\int z\, dx\, dy\, dz \tag{2}$$

Now the integral with respect to dx and dy is just finding the area of circles of radius r in the $x - y$ plane, meaning

$$\int dx\, dy = \pi r^2 = \pi \frac{R^2 z^2}{h^2} \tag{3}$$

Thus,

$$Z = \frac{\rho\pi R^2}{Mh^2}\int_0^h z^3\, dz = \frac{\rho\pi R^2 h^4}{4Mh^2} = \frac{\rho\pi R^2 h^2}{4M} \tag{4}$$

Since the volume of a cone is $V = \frac{1}{3}\pi R^2 h$, we can substitute for the mass $M = \rho V = \frac{1}{3}\rho\pi R^2 h$, which gives

$$Z = \frac{\rho\pi R^2 h^2}{4} \cdot \frac{3}{\rho\pi R^2 h} = \frac{3}{4}h \tag{5}$$

So the center of mass is 1/4 the way down from the base.

2 Angular momentum of a single particle

Recall for a single particle that $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position of the particle relative to the origin. One difference with linear momentum is that since $\mathbf{r}$ depends on the choice of origin, so does $\mathbf{l}$. Therefore, to be strictly correct we should specify the angular momentum about a particular point, but usually this is clear from the stated choice of origin.

Figure 1: *Position, momentum, and angular momentum vectors.*

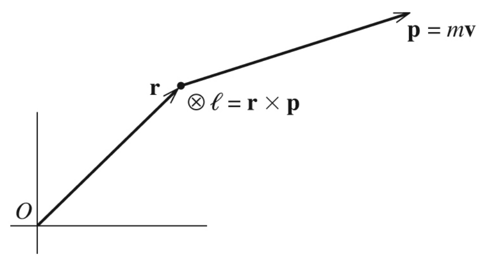


Figure 3.5
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What is the time rate of change of angular momentum?

$$\dot{\mathbf{l}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = [\dot{\mathbf{r}} \times (m\dot{\mathbf{r}})] + [\mathbf{r} \times \dot{\mathbf{p}}] = m[\dot{\mathbf{r}} \times \dot{\mathbf{r}}] + [\mathbf{r} \times \dot{\mathbf{p}}] \tag{6}$$

For the last step, we noted that $p = m\dot{\mathbf{r}}$. Since $\mathbf{a} \times \mathbf{a} = 0$ for any vector $\mathbf{a}$, the first term is zero and we have

$$\dot{\mathbf{l}} = \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} \equiv \mathbf{\Gamma} \tag{7}$$

Here we used $\dot{\mathbf{p}} = \mathbf{F}$. The quantity $\mathbf{\Gamma} \equiv \mathbf{r} \times \mathbf{F}$ is the net torque on the particle (which often goes by $\boldsymbol{\tau}$ or $\mathbf{N}$ in other texts).

- We have established that the time rate of change of angular momentum about the origin is the torque about the origin
- The analog of $\dot{\mathbf{p}} = \mathbf{F}$ for linear momentum is $\dot{\mathbf{l}} = \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma}$ for rotation
- Can often chose an origin such that $\mathbf{\Gamma} = 0$, meaning $\mathbf{l}$ is constant. For example, all forces point toward or away from the origin.
- Central forces in particular often have this possibility.

For instance, say we have a planet orbiting a sun as shown below. If we choose the origin as the sun's position, then the radial vector $\mathbf{r}$ pointing to the planet's position is directed along the same line as the force between planet and sun. That means $\dot{\mathbf{l}} = \mathbf{r} \times \mathbf{F} = 0$, and thus $\mathbf{l}$ is conserved. If $\mathbf{l}$ is conserved, then $\mathbf{r} \times \mathbf{p}$ is constant. For that to be the case, $\mathbf{r}$ and $\mathbf{p}$ must lie in a plane. That means that we can treat the orbit of planets as a two-dimensional problem, which is nice.

Figure 2: *Planet orbiting a sun.*

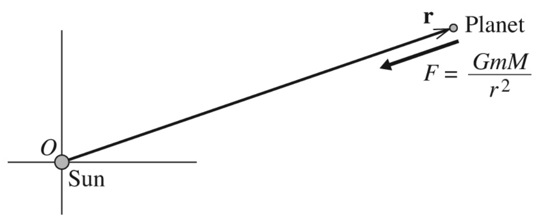

Figure 3.6
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3 Kepler's second law

Figure 3: *Setup for Kepler's second law*

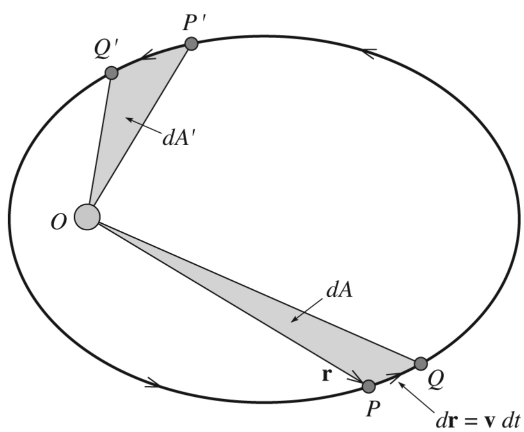

Figure 3.7
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In short, Kepler's second law states that if (P, Q) and (P', Q') are separated by equal time intervals dt, then $dA = dA'$. We can demonstrate it relatively easily. Recall that the area of a triangle with two sides $\mathbf{a}$ and $\mathbf{b}$ can be written as

$$A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \tag{8}$$

From the figure above, we can see

$$dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}\, dt| \tag{9}$$

Noting that $v = p/m$, and dividing by dt,

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r} \times \mathbf{p}| = \frac{\mathbf{l}}{2m} \tag{10}$$

Since $\mathbf{l}$ is conserved about the sun (central force, $\mathbf{\Gamma} = 0$), then $\mathbf{l}$ and dA/dt are constant - equal areas swept out in equal times. We could also go at this another way:

$$\frac{dA}{dt} = \frac{\text{area of arc}}{dt} = \frac{\frac{1}{2}r^2\, d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r^2\omega \tag{11}$$

With $l = mr^2\omega$ for a particle, we get the same result $dA/dt = l/2m$. We can also note that $\omega \propto 1/r^2$, which means that if the orbital distance is halved, the speed goes up by a factor of 4. That is, if r at P' is half of r at P, then ω at P' is 4 times ω at P.

4 Angular momentum for several particles

Say we have N particles labeled $\alpha = 1, 2, \ldots, N$. Each has angular momentum

$$\mathbf{l}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{12}$$

The total angular momentum of the collection is then

$$\mathbf{L} = \sum_{\alpha=1}^{N} \mathbf{l}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{13}$$

Now how does $\mathbf{L}$ change with time? Using our previous result for $\dot{\mathbf{l}}$,

$$\dot{\mathbf{L}} = \sum_{\alpha} \dot{\mathbf{l}}_\alpha = \sum_{\alpha} \mathbf{r} \times \mathbf{F}_\alpha \tag{14}$$

The change in total angular momentum is just the net torque, as we already knew. But, there are two types of forces on α: external forces, and forces due to interactions with other particles.

Explicitly,

$$\text{net force on } \alpha = \mathbf{F}_\alpha = \sum_{\beta\neq\alpha} \mathbf{F}_{\alpha\beta} + \sum_{\alpha} \mathbf{F}_\alpha^{\text{ext}} \tag{15}$$

Here $\mathbf{F}_{\alpha\beta}$ represents an interaction between particle α and particle β (summing over $\beta \neq \alpha$ means we do not allow self-interaction of a particle), and $\mathbf{F}_\alpha^{\text{ext}}$ represents forces on particle α due to external agents. Going back to angular momentum,

$$\dot{\mathbf{L}} = \sum_{\alpha} \sum_{\beta\neq\alpha} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \sum_{\alpha} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} \tag{16}$$

Going back to our discussion of linear momentum, along similar lines we can split the sum of interaction forces into two separate sums, pairing each term $\alpha\beta$ with a corresponding term $\beta\alpha$. That is, rather than taking a single particle α, counting its interaction with all other particles β, and then adding the result for all such α, we find the interaction of α with β and β with α and adjust the limits of the sum so we don't double count any interactions.

$$\sum_{\alpha} \sum_{\beta\neq\alpha} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} = \sum_{\alpha} \sum_{\beta>\alpha} \left(\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \mathbf{r}_\beta \times \mathbf{F}_{\beta\alpha}\right) \tag{17}$$

If Newton's third law holds, then for every $\mathbf{F}_{\alpha\beta}$ there exists $\mathbf{F}_{\alpha\beta} = -\mathbf{F}_{\beta\alpha}$ - for every interaction, there is an equal and opposite reaction. If this is the case, then we can write the right most sum as

$$\sum_{\alpha} \sum_{\beta\neq\alpha} \left(\mathbf{r}_\alpha - \mathbf{r}_\beta\right) \times \mathbf{F}_{\alpha\beta} \tag{18}$$

Now what is the vector $\mathbf{r}_\alpha - \mathbf{r}_\beta \equiv \mathbf{r}_{\alpha\beta}$? It is a vector pointing from β to α, as shown below.

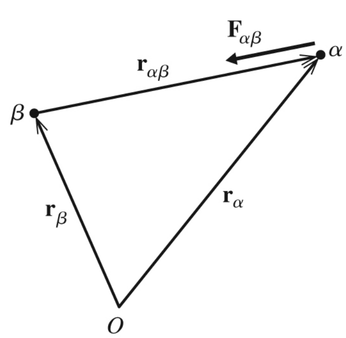

Figure 3.8
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Now, if we further assume that all the interactions are the result of central forces, then $\mathbf{r}_{\alpha\beta}$ and $\mathbf{F}_{\alpha\beta}$ are parallel (or antiparallel), and their cross product must be zero. That gives us the result

$$\dot{\mathbf{L}} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}} \tag{19}$$

The net torque is just the sum of the individual torques on each α due to *external* forces. If the net torque is zero $\mathbf{\Gamma}^{\text{ext}} = 0$ (say, if all external forces are zero or act only on the center of mass), then $\mathbf{L}$ is constant. This relied on two key points:

- forces are central (e.g., gravity, electrostatics)
- Newton's 3rd law holds

These are not serious restrictions, and the result is most of the time valid. Somewhat remarkably, the same result holds about the center of mass *even if* the center of mass is accelerating (non-inertial reference frame). You'll prove this in one of the homework problems (3.37).

$$\frac{d}{dt}\mathbf{L}\,(\text{about CM}) = \mathbf{\Gamma}^{\text{ext}}\,(\text{about CM}) \tag{20}$$

5 Moments of inertia

We'll revisit moments of inertia in Ch. 10. Recall that if the rotation axis is the z axis, then $L_z = I\omega$, where I is the moment of inertia (the second moment of the matter distribution). We can find I about a given axis for a given shape with

$$I = \sum_{\text{shape}} m_{\alpha} r_{\alpha}^{2} \Longrightarrow \int_{\text{shape}} r^{2}\, dm \tag{21}$$