University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 8: Ch. 4.1-4

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1 Energy

As you know kinetic energy is $T = \frac{1}{2}mv^2$. Let's say we have a particle going from $\mathbf{r}_1$ to $\mathbf{r}_1 + d\mathbf{r}$ as shown below. How does T change with time?

Figure 1: *Particle on a path*

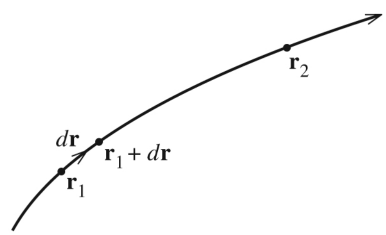


Figure 4.1
Taylor CLASSICAL MECHANICS

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}m(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) = m\dot{\mathbf{v}}\cdot\mathbf{v} \tag{1}$$

But we know $m\dot{\mathbf{v}} = \mathbf{F}$, where $\mathbf{F}$ is the net force on the particle, so

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \text{or} \qquad dT = \mathbf{F}\cdot d\mathbf{r} \tag{2}$$

This makes it clear that $\mathbf{F}\cdot d\mathbf{r} = dT = dW$, the work done in moving the particle from $\mathbf{r}_1$ to $\mathbf{r}_1 + d\mathbf{r}$. This is the work-energy theorem, albeit only for infinitesimal displacements. The generalization to finite displacements is easy enough, we sum over all $d\mathbf{r}$ taking the particle from $\mathbf{r}_1$ to $\mathbf{r}_2$.

$$\Delta T \equiv T_2 - T_1 = \sum_{\mathbf{r}_1\to\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} \qquad \text{take limit } d\mathbf{r}\to 0 \tag{3}$$

$$\Delta T = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = W_{1\to 2} \tag{4}$$

This is the work-energy theorem. The integral in question is a *line integral*, taken over a specific path from point 1 to point 2. In one dimension this gives us the usual result. In more than one dimension, we have to be careful about which path we specify. If the net force involved is conservative, it turns out the work done will be path-independent, and only the endpoints matter.

Went through Example 4.1 in class / example line integral over three different paths.

Note that the $\mathbf{F}$ above is the *net force*, $\mathbf{F} = \sum_i \mathbf{F}_i$. Since we can interchange summation and integration, we have two options: find the net force, and then find the work done, or find the work done by each force W_i and sum them.

$$W_{1\to 2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sum_i \mathbf{F}_i \cdot d\mathbf{r} = W_{1\to 2} = \sum_i \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_k \cdot d\mathbf{r} = \sum_i W_i \tag{5}$$

2 Potential Energy and Conservative Forces

1. for a conservative force, $F = F(r)$, i.e., the force depends only on position (not velocity or time for example). OK for gravity, electrostatics. Not OK for air resistance, friction, magnetic force, or time-varying electric fields

2. for a conservative force, $W_{1\to 2}$ is independent of path. Clearly not OK for friction since work depends on the length of the path rather than the displacement. Clearly OK for gravity since W depends only on the change in vertical coordinate

Figure 2: *Particle on three different paths. The work done in getting from start to end is the same along all paths for a conservative force. Can you see why this is not the case for friction or air resistance?*

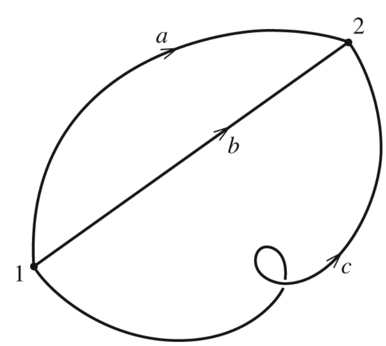


Figure 4.3
Taylor CLASSICAL MECHANICS

If the force is conservative, total mechanical energy is conserved

$$E = KE + PE = T + U(\mathbf{r}) \tag{6}$$

Defining U? At some reference point $\mathbf{r}_o$, $U(\mathbf{r}_o) = 0$, such that

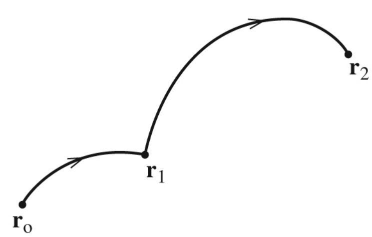
$$U(\mathbf{r}) = -W(\mathbf{r}_o \to \mathbf{r}) \equiv -\int_{\mathbf{r}_o}^{\mathbf{r}} F(\mathbf{r}') \cdot d\mathbf{r}' \tag{7}$$

1. if W depends on path, clearly the integral is not well defined and so is U. No meaningful potential energy function.
2. U is minus the work that would be done in moving the particle from $\mathbf{r}_o$ to $\mathbf{r}$

For example, a charge in a static electric field $\mathbf{E} = E_o\,\hat{\mathbf{x}}$, giving rise to a force $\mathbf{F} = q\mathbf{E} = qE_o\,\hat{\mathbf{x}}$:

$$W_{1\to 2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 qE_o\hat{\mathbf{x}} \cdot d\mathbf{r} = qE_o \int_1^2 dx = qE_o(x_2 - x_1) \tag{8}$$

Figure 3: *Particle on a path*



Since the result depends only on the coordinates of the endpoint, the path taken from 1 to 2 is irrelevant, and the electric force must be conservative. Making use of the free choice $\mathbf{r}_o = 0$, we can then define a potential energy function that gives the energy that would be required to move from $\mathbf{r}_o$ to a new position $\mathbf{r}$:

$$U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -qE_o x \tag{9}$$

Now we can relate work and potential energy. From the figure above, we would like to find the work done in going from 0 to 2 under the influence of a conservative force. Since the work is path-independent, we can split this into two pieces: the work done in going from 0 to 1 plus the work done in going from 1 to 2:

$$\begin{aligned} & W_{0\to2} = W_{0\to1} + W_{1\to2} && (10) \\ \implies \quad & W_{1\to2} = W_{0\to2} - W_{0\to1} = -\left[U(\mathbf{r}_2) - U\mathbf{r}_1\right] = -\Delta U && (11) \end{aligned}$$

But we also know that $\Delta T = W_{1\to2}$, so that establishes

$$\Delta T = -\Delta U \qquad \text{or} \qquad \Delta(T+U) = 0 \tag{12}$$

That is, mechanical energy $T + U$ is conserved when only conservative forces are present. What if we have several conservative forces? We just handle the potential energy change for each separately, for example:

$$\begin{aligned} \Delta T = W_{\text{gravity}} + W_{\text{spring}} &= -(\Delta U_{\text{gravity}} + \Delta U_{\text{spring}}) && (13) \\ E = T + U_{\text{gravity}} + U_{\text{spring}} &= \text{constant} && (14) \end{aligned}$$

What about non-conservative forces? We can still keep an energy budget, since we can still calculate the work done by non-conservative forces. The only issue is that we can't define a potential energy for the non-conservative force, and the mechanical energy is not constant. The external force doing work can either add or subtract mechanical energy from the system.

$$\begin{aligned} & \Delta T = W = W_{\text{consv.}} + W_{\text{non-consv.}} = -\Delta U + W_{\text{non-consv.}} && (15) \\ \implies \quad & \Delta E = \Delta(T+U) = W_{\text{non-consv.}} && (16) \end{aligned}$$

So bookkeeping still works. For friction, for example, if the particle travels a path of distance L, the work done is $W_{\text{non-consv.}} = -F_{\text{friction}}L$ (negative since force and displacement are always in opposite directions). That would lead to a decrease of mechanical energy of the system $\Delta < 0$.

3 Force and Potential Energy

Recall

$$W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz \tag{17}$$

But we know this also equates to the potential energy change:

$$W(\mathbf{r} \to \mathbf{r}+d\mathbf{r}) = -dU = -\left[U(\mathbf{r}+d\mathbf{r}) - U(\mathbf{r})\right] = -\left[U(x+dx, y+dy, z+dz) - U(x,y,z)\right] \quad (18)$$

If we note that for a function f the definition of the derivative gives

$$df = f(x+dx) - f(x) = \frac{df}{dx}dx \quad (19)$$

Then for a function of 3 variables like U we can write

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz \quad (20)$$

$$\Longrightarrow \qquad W = -\left[\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz\right] \quad (21)$$

Comparing our equations, we must have

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \quad (22)$$

Thus, the force is derivable from the potential energy just as we developed the potential energy from the force. In fact, the force is the negative gradient of the potential.

$$\mathbf{F} = -\left(\hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z}\right) = -\boldsymbol{\nabla} U \quad (23)$$

Think about this in terms of gravity: if the gradient of a surface is positive (uphill), potential energy is lower downhill. Also note that $df = \boldsymbol{\nabla} f \cdot d\mathbf{r}$ in this notation for functions of several variables.

4 Establishing path independence

Let C be a simple closed path (one that starts and ends on the same point and doesn't cross itself) which is also a boundary of the surface S. Stoke's theorem states

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} \quad (24)$$

(The symbol $\oint$ just means the integral is over a closed path that starts and ends on the same point.) For a conservative force, the work done around a closed path (the first term) must be zero. Think of it this way: for a conservative force the work is a function of coordinates alone, so $W(a \to b) = -W(b \to a)$. Therefore, going from a to b and then from b to a along any closed path has net work zero.

If the first term is zero, the second term must also be zero, so we require $\boldsymbol{\nabla} \times \mathbf{F} = 0$ for a conservative force – a much simpler way to test if a force is conservative. This also means that conservative force fields are *irrotational* (which spells trouble for fluids). In cartesian coordinates the curl is easy enough to calculate, given a vector function $\mathbf{F}(x, y, z)$, you can think of it as the determinant of an easy-to-remember 3×3 matrix.

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}} \tag{25}$$

You can look up the more complicated expressions for other coordinate systems. You can also prove that $\boldsymbol{\nabla} \times \mathbf{F} = 0$ for any central force of the form $\mathbf{F} = \mathbf{F}(\mathbf{r})$, which we will cover later in Ch. 4. This means Coulomb's law and Newton's law of Universal Gravitation are both conservative. We can also note that since for any f it is true that $\nabla \times \nabla f = 0$, any force field which can be derived from a potential energy $\mathbf{F} = -\boldsymbol{\nabla} U$ automatically has zero curl and is conservative.

We now have three ways to show a force is conservative, and each one implies the other. Show that any one is true, and the other two statements are also true, and your force is conservative.

$$\boldsymbol{\nabla} \times \mathbf{F} = 0 \quad \Longleftrightarrow \quad \oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad \Longleftrightarrow \quad \mathbf{F} = -\boldsymbol{\nabla} U \tag{26}$$