University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Lecture 8: Ch. 4.5-7

12 Sept 2018

1 Time-dependent potential energy

Sometimes we need to consider time-dependent forces, $\mathbf{F}(\mathbf{r},t)$. The force may still satisfy $\boldsymbol{\nabla}\times\mathbf{F}=0$. However, it does not satisfy our first condition for a conservative force, viz., the force should be a function of *position alone*. However, can still define $U(\mathbf{r},t)$ such that $\mathbf{F}=-\boldsymbol{\nabla}U$ since $\boldsymbol{\nabla}\times\mathbf{F}=0$. The consequence of a time-varying U is that mechanical energy will no longer be conserved.

Consider one concrete example from the text, a single charge q near a van de Graaff generator whose conducting globe has built up a charge Q that is leaking away to the surrounding air, so $Q=Q(t)$.

Figure 1: *Test charge q near a van de Graaff generator.*

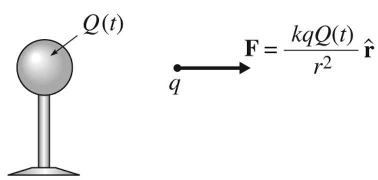

Figure 4.8
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The force on q is now explicitly time-dependent because Q is: $\mathbf{F}=\frac{kqQ(t)}{r^2}\,\hat{\mathbf{r}}$. Since F is still the Coulomb force, it still satisfies $\boldsymbol{\nabla}\times\mathbf{F}=0$, meaning we can write F as the gradient of a scalar function, i.e., the potential energy:

$$U(\mathbf{r},t)=-\int\limits_{\mathbf{r}_o}^{\mathbf{r}}\mathbf{F}(\mathbf{r}',t)\cdot d\mathbf{r}' \tag{1}$$

$$\mathbf{F}(\mathbf{r},t)=-\boldsymbol{\nabla}U(\mathbf{r},t) \tag{2}$$

So far so good! But what are the consequences? Let's look at the time variation of kinetic energy again.

$$dT=\frac{dT}{dt}\,dt=(m\dot{\mathbf{v}}\cdot\mathbf{v})\;dt=\mathbf{F}\cdot d\mathbf{r} \tag{3}$$

Since $U(\mathbf{r}, t) = U(x, y, z, t)$, we can also find dU.

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz + \frac{\partial U}{\partial t}\,dt = -\mathbf{F} \cdot d\mathbf{r} + \frac{\partial U}{\partial t}\,dt \tag{4}$$

The first three terms in the second expression are $\boldsymbol{\nabla} U \cdot d\mathbf{r}$. From our previous results, we know that $\boldsymbol{\nabla} U \cdot d\mathbf{r} = -\mathbf{F} \cdot d\mathbf{r}$, so when we add $dU + dT$, the first three terms will cancel!

$$dT + dU = d(T + U) = \mathbf{F} \cdot d\mathbf{r} - \mathbf{F} \cdot d\mathbf{r} + \frac{\partial U}{\partial t}\,dt = \frac{\partial U}{\partial t}\,dt \tag{5}$$

$$\tag{6}$$

Clearly, mechanical energy is only conserved when $\partial U/\partial t = 0$, that is, when U is independent of time.

Why is that? Back to the van de Graaff example.

- hold q stationary while charge Q leaks away
- T doesn't change for q, but U does because Q is decreasing
- $\Longrightarrow T + U \neq$ constant **for q**
- problem: we have an open system, which excludes Q.
- gain of thermal energy for surrounding air as discharge heats it up
- no problem if system $= q + Q +$ air

U is dependent on time when mechanical energy is transferred to another kind or to other bodies outside the system of interest. Can alleviate this in some cases by defining the system more cleverly, or we have to account for the time-dependent work done by external agents that transfer energy into or out of our system.

2 One Dimensional Systems

Need not be straight line paths, just a situation where the only directions are forwards $\rightarrow$ and backwards $\leftarrow$. Can be a curved path, or any situation where 1 variable is enough to describe the motion (e.g., pendulum). In this case

$$W(x_1 \rightarrow x_2) = \int_{x_1}^{x_2} F_x(x)\,dx \tag{7}$$

If F is conservative, (i) F depends only on the coordinate x, and (ii) W is independent of path. In 1D, statement (i) implies statement (ii), so (ii) is a redundant condition!

Figure 2: *Two ways to get from A to B in 1D*

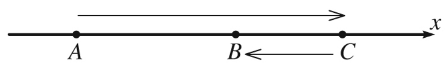

Figure 4.9

In the figure above, we can go from A to B directly, or via point C. The two paths we can label “AB” and “ABCB”. What is the work done on the more complicated ABCB path? We can break it up into segments. Since F is conservative, the work depends only on the starting and ending coordinates.

$$W(ABCB) = W(AB) + W(BC) + W(CB) \tag{8}$$

If F depends only on position, then it must be true that $W(BC) = -W(CB)$, which gives

$$W(ABCB) = W(AB) \tag{9}$$

So long as F depends only on position in 1D, the work on *any* path from A to B is the same. Think about the force of gravity - you already know the work depends only on the change in y coordinate and nothing else. Rotate the figure above by 90° and it is a problem about a ball tossed in the air - the ball has the same energy when it first reaches B as it does when it eventually comes back down to point B (in the absence of air resistance at least, but that would be a force that does not only dependent on position anyway).

3 Graphs of U in 1D

We already know some examples of U in one dimension: for a spring, $U(x) = \frac{1}{2}kx^2$. The relation to force is also simple in 1D: $F_x = -dU/dx$. Making plots of potential energy can be useful in figuring out what the system is doing.

Figure 3: *Potential landscape in 1D*

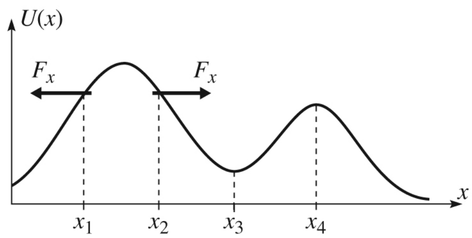

Figure 4.10
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For one, the force is "downhill" on the $U(x)$ graph. In the figure above, at points x_1 and x_2 the force is downhill, to the left and right respectively. At both points the force tries to move the particle to a lower U.

At points x_3 and x_4, $dU/dx = 0$, so we have an *equilibrium point.* At x_3, $d^2U/dx^2 > 0$, the curve is concave up, and we have a stable equilibrium. For any displacement, the force acts in opposition to bring the system back to equilibrium. In general the particle will oscillate about x_3 once displaced.

At x_4, $d^2U/dx^2 < 0$, the curve is concave down, and the equilibrium is unstable. Any tiny displacement leads to a force in the same direction, moving away from equilibrium.

What if the particle is moving? Then the particle has some fixed energy $E = T + U$. That means the maximum value of U is restricted to $U = E$, at which point $T = 0$ and the particle must come to a stop.

Figure 4: *A moving particle in 1D*

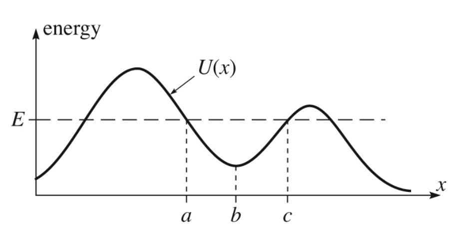

Figure 4.11
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From the graph above, we can see that the particle reaches maximum U at points a and c, where it must have $T = 0$. The particle cannot go outside the region $x \in [a, b]$ because it doesn't have enough energy to go higher. So $x < a$ and $x > c$ are inaccessible and the particle is trapped. If displaced it will oscillate about point c.

What if the particle has an energy E higher than the tallest "hill"? It will continue in the same direction as its initial velocity indefinitely.

One example would be a molecular bond, like H bound to Cl in HCl (Cl is so much heavier than H we can consider its position to be a fixed point that the H atom moves around). Note the figure below.

Figure 5: *Radial potential for a molecule*

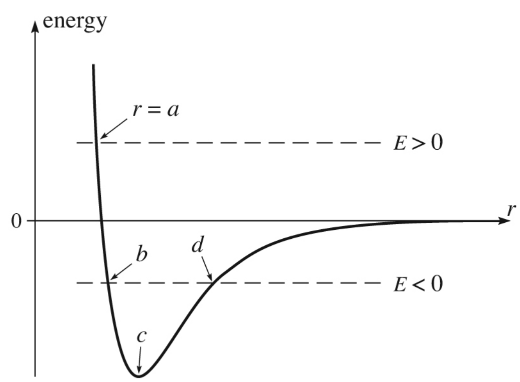

Figure 4.12
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- As $r \to 0$, $U \to \infty$, and the atoms repel. They can never touch, since the energy required would be infinite.
- If $E > 0$, the H atom can escape to ∞ and the molecule breaks
- If $E > 0$ you can bring the H atom in to a radius $r = a$, then it is repulsed and escapes back to ∞
- If $E < 0$ H is bound to Cl between r_b and r_d, with an equilibrium position at r_c.
- Any concave upward potential is approximately parabolic for small displacements from equilibrium, which means the H atom acts like a harmonic oscillator for small displacements from equilibrium - the molecule has a characteristic vibrational frequency determined by the masses and bond strength!

4 Complete solution of motion in 1D

Remarkably, in 1D we can completely solve the motion from potential alone. Start with energy conservation and solve for $\dot{x}$.

$$T = \frac{1}{2}m\dot{x}^2 = E - U(x) \tag{10}$$

$$\dot{x}(x) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)} \tag{11}$$

We explicitly show here that $\dot{x}$ is a function of position x. Noting that $\dot{x} = dx/dt$, we know $dt = dt/\dot{x}$. This will let us separate and integrate, as on the homework.

$$\int_0^t dt' = t = \int_{x_o}^{x} \frac{dx'}{\dot{x}'} \tag{12}$$

$$t = \int_{x_o}^{x} \frac{dx'}{\dot{x}'(x')} = \sqrt{\frac{m}{2}} \int_{x_o}^{x} \frac{dx'}{\sqrt{E - U(x')}} \tag{13}$$

The integral can always be solved numerically given some initial conditions, and you showed in the homework that it works analytically for the simple harmonic oscillator. So in fact the whole motion is solved for any 1D problem, so long as we can specify a potential $U(x)$.

We can check the result for free fall. Let $x_o = v_o = 0$, i.e., a particle dropped from rest at the origin. Then $U(x) = -mgx$ and $E = 0$, so

$$t = \sqrt{\frac{m}{2}} \int_0^x \frac{dx'}{\sqrt{-U(x')}} = \int_0^x \frac{dx'}{\sqrt{2gx'}} = \sqrt{\frac{2x}{g}} \qquad \Longrightarrow \qquad x = \frac{1}{2}gt^2 \tag{14}$$

5 Curvilinear 1D Systems

Let s be the distance covered along the path. Clearly the speed is then $\dot{s}$, so $T = \frac{1}{2}m\dot{s}^2$.

Figure 6: *A curvilinear 1D path*

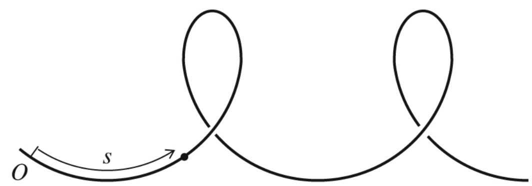

Figure 4.13
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The net force is more complicated, there are 2 parts: (1) acceleration along the path (speeding up/slowing down), and (2) acceleration perpendicular to the path that allows turning but doesn't change $|\mathbf{v}|$. It is easy to see that long the path,

$$F_{\text{tang}} = m\ddot{s} = -\frac{dU}{ds} \tag{15}$$

If you know the equation for the path and the potential energy function, you know the force. $E = T + U(s)$ is constant. There must also be a normal component to the force to cause turning.

- F_c is a force of constraint and fully specified by s and $\dot{s}$
- F_c does no work since it is by construction perpendicular to the path heading.

You can show for some path described by $y(x)$ in 2D that

$$\Delta s = \int_i^f \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{16}$$

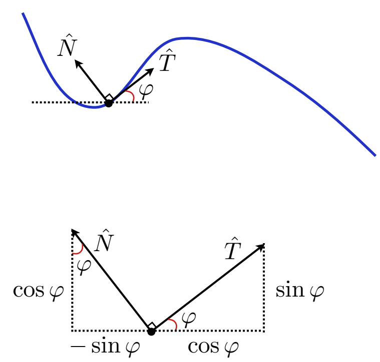
$$\tag{17}$$

At any given point along the path, there are two directions: normal and tangential, and they both have unit vectors:

$$\hat{s} = \hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{18}$$

$$\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}/ds}{|d\hat{\mathbf{T}}/ds|} \tag{19}$$

Figure 7: *Tangential and normal unit vectors along a path, and the inclination angle φ.*



Basically, $\hat{\mathbf{T}}$ points in the direction you are heading, $\hat{\mathbf{N}}$ points in the direction your turning, and the two are perpendicular at any point on the curve specifying s. The full acceleration is

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\mathbf{v}|^2\,\hat{\mathbf{N}} = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{|\mathbf{v}|^2}{R}\,\hat{\mathbf{N}} \equiv a_N\,\hat{\mathbf{T}} + a_T\,\hat{\mathbf{N}} \tag{20}$$

Here κ is the curvature of the path at some point. If the path is described by $y(x)$:

$$\kappa = \frac{\left|\dfrac{d^2y}{dx^2}\right|}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{21}$$

One can also develop an equivalent expression for curvature based on the parametric expression for the same path, $y(t)$ and $x(t)$.

$$\kappa = \frac{\left|\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}\right|}{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]^{3/2}} \tag{22}$$

The generalized radius of curvature is $R = \frac{1}{\kappa}$, and for a circle this is just the radius. For a more complicated curve, you can imagine R is the radius of a circle that would best fit the curve at that point. Back to the expression for acceleration (Eq. 20): the first term ($\hat{\mathbf{T}}$ component) is just the rate at which your speed is changing along the path, and the second term ($\hat{\mathbf{N}}$ component) is a generalization of the centripetal force from circular motion, $a_c = v^2/R$.

Qualitatively? The $\hat{\mathbf{N}}$ component says for a given speed the force required to turn is higher for sharper curves, and at a given curvature higher for a higher speed.

6 Making things one dimensional

Can make many systems that look complicated into 1D systems. Example - Atwood's machine: for a fixed length rope, only one coordinate is needed.

Figure 8: *Atwood's machine: only the coordinate x is needed to determine the motion of the system, so this is a 1D problem.*

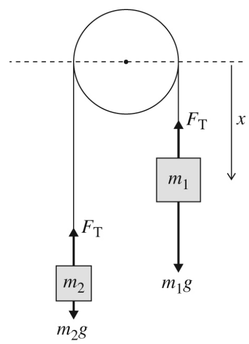


Figure 4.15
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Example 4.7 from the book - block balancing on a cylinder: the single coordinate θ determines the position of the block, so this is actually a 1D problem!

Figure 9: *Box balancing on a cylinder*

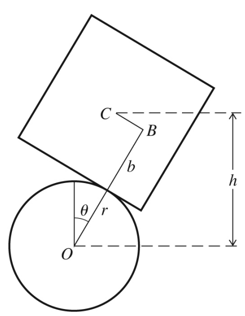

Figure 4.14
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The vertical displacement h has three parts: the vertical part of the cylinder radius $mgr\cos\theta$, the vertical part of the distance b which is $mgb\cos\theta$ and the vertical distance from B to C, which is the arclength the system is displaced through from $\theta = 0$, or $r\theta \sin\theta$. Thus,

$$U = mgh = mgr\cos\theta + mbg\cos\theta + r\theta\sin\theta = mg\left[(r+b)\cos\theta + r\theta\sin\theta\right] \tag{23}$$

You can find $dU/d\theta$ easily enough to figure out the equilibrium position:

$$\frac{dU}{d\theta} = mg\left[r\theta\cos\theta - b\sin\theta\right] \tag{24}$$

$dU/d\theta$ vanishes at the equilibrium position $\theta = 0$, just as you would expect. What is the stability of the equilibrium? The sign of the second derivative tells us:

$$\frac{d^2U}{d\theta^2} = mg\,(r-b) \tag{25}$$

So long as the cube is smaller than the cylinder ($r > b$) the equilibrium is stable; if the cube is larger ($r < b$) it is unstable and the cube will fall off with the slightest perturbation. Pretty much what you would expect.