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PH 125 / LeClair Spring 2009

Motion along a curved path

Early on, we learned how to describe two-dimensional (2D) motion in terms of a position vector $\vec{\mathbf{r}}(t)$ that gives a vector pointing from our coordinate system origin ($\mathcal{O}$) to the current position of an object. In terms of that position vector, we defined velocity and acceleration vectors as successive time derivatives of the position vector with respect to time. Any of those three vectors could be defined in terms of components in our chosen coordinate system. For instance, if we choose an $x-y$ cartesian coordinate system, the position, velocity, and acceleration vectors have x and y components. Using unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ to represent unit-length movements along the x and y axes, respectively, we can write both general (coordinate-free) and specific (component form) expressions for all three vectors:

$$\begin{aligned}
\vec{\mathbf{r}}(t) &= x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} \\
\vec{\mathbf{v}}(t) &= \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} = v_x\,\hat{\boldsymbol{\imath}} + v_y\,\hat{\boldsymbol{\jmath}} \\
\vec{\mathbf{a}}(t) &= \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{a}}}{dt^2} = \frac{d^2x}{dt^2}\,\hat{\boldsymbol{\imath}} + \frac{d^2y}{dt^2}\,\hat{\boldsymbol{\jmath}} = \frac{dv_x}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dv_y}{dt}\,\hat{\boldsymbol{\jmath}} = a_x\,\hat{\boldsymbol{\imath}} + a_y\,\hat{\boldsymbol{\jmath}}
\end{aligned} \tag{1}$$

Within a particular coordinate system, we have a set of parametric equations for the position. In this example, the x and y motion are decoupled, and the time evolution of each coordinate is separately described by $x(t)$ and $y(t)$.

1 Distance covered along an arbitrary path

We left open some seemingly straightforward questions, however. How far does an object travel along the path described by $\vec{\mathbf{r}}(t)$? What is its heading, or direction, at any particular instant? More precisely, we can easily calculate the *displacement* $\Delta\vec{\mathbf{r}}$ between any two times t_i and t_f, and we can easily calculate the angle of the particle's path with respect to the x axis at any instant:

$$\begin{aligned}
\Delta\vec{\mathbf{r}} &= \vec{\mathbf{r}}(t_f) - \vec{\mathbf{r}}(t_i) \\
\theta_i &= \tan^{-1}\left[\frac{y(t_i)}{x(t_i)}\right] = \tan^{-1}\left[\frac{v_y(t_i)}{v_x(t_i)}\right]
\end{aligned} \tag{2}$$

However, we do not have a way to calculate the *total* distance covered in getting from $\vec{\mathbf{r}}_i$ to $\vec{\mathbf{r}}_f$, nor do we have a nice way of describing how much an object's orientation changes with time. Specifically, we would like to know the actual length of the path covered, not just the net distance between two points, and we would like to know how “curvy" the path is, how much orientation changes per unit length.

Let us take a fairly arbitrary example, a particle traveling on a generic path s described by the position vector $\vec{\mathbf{r}}(t)$, as shown below. Now let's look at distance between two positions separated by an infinitesimal time step dt, *viz.*, $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}(t+dt)$. In that time interval dt, our particle will have covered some tiny length of the path ds.

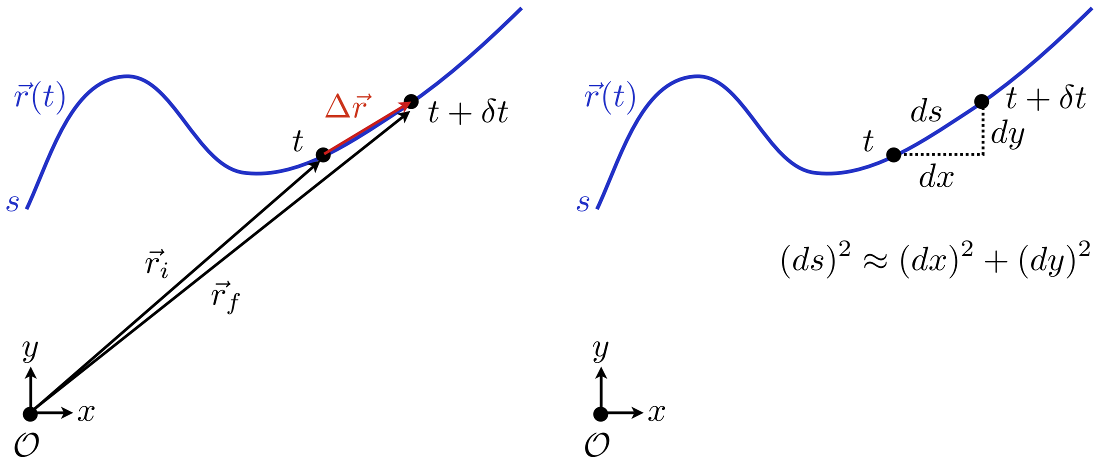

Figure 1: *A general path s described by the position vector $\vec{\mathbf{r}}(t)$.* **left:** *The displacement between two successive positions $\vec{\mathbf{r}}_i = \vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}_f = \vec{\mathbf{r}}(t+dt)$.* **right:** *The actual distance covered along the path s between t and $t+\delta t$ is ds. If dt is very small, ds is approximately a line segment.*

The *displacement* between the successive positions $d\vec{\mathbf{r}}$ is easily calculated, as above. Let us say over the time interval dt the particle's x position changes by an amount dx, and its y position changes by dy. The displacement is then

$$\begin{aligned}
\vec{\mathbf{r}}_i &= \vec{\mathbf{r}}(t) = x_i\,\hat{\boldsymbol{\imath}} + y_i\,\hat{\boldsymbol{\jmath}} \\
\vec{\mathbf{r}}_f &= \vec{\mathbf{r}}(t+dt) = (x_i + dx)\;\hat{\boldsymbol{\imath}} + (y_i + dy)\;\hat{\boldsymbol{\jmath}} \\
d\vec{\mathbf{r}} &= \vec{\mathbf{r}}(t+dt) - \vec{\mathbf{r}}(t) = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}
\end{aligned} \tag{3}$$

This is not the same as the actual distance covered along the path s. The displacement represents the shortest possible distance between $\vec{\mathbf{r}}(t)$ and $\vec{\mathbf{r}}(t+\delta t)$, but the actual distance covered depends on s itself, and how much it deviates from a straight line between t and $t+dt$. If we make dt small enough, however, the tiny segment of the path covered over that time interval ds will be *approximately* a line segment. In that case, we can estimate the path length ds from the displacement. If we let $dt \to 0$, the relationship is "exact."

$$(ds)^2 = |d\vec{\mathbf{r}}|^2 = (dx)^2 + (dy)^2 \tag{4}$$

From now on, we will assume that dt is infinitesimally small. This expression gives us the path length ds over an infinitesimally small time step dt, which is nothing more than finding the distance between points (x, y) and $(x + dx, y + dy)$, provided dx and dy are small enough. We could put this relationship another way as well:

$$\left|\frac{d\vec{\mathbf{r}}}{ds}\right| = 1 \tag{5}$$

Basically, the change in displacement is identical to the change in distance for very small displacements. This does lead us to an important result, however: we can define a unit vector which always points along the direction of the instantaneous displacement, *a unit vector tangent to the curve.*

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \frac{dx}{ds}\,\hat{\boldsymbol{\imath}} + \frac{dy}{ds}\,\hat{\boldsymbol{\jmath}} \tag{6}$$

This unit vector $\hat{\mathbf{T}}$ – and it must be a unit vector from Eq. 5 – always gives us the direction of the particle's trajectory at a given time or position, which will be very useful shortly.

Given the distance ds covered in a time interval dt, we can also find the particle's speed and velocity. If the particle follows the position vector

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} \tag{7}$$

then the velocity is easily found:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} \tag{8}$$

The *speed* of the particle can be calculated either by finding $|\vec{\mathbf{v}}|$, or dividing the distance ds by the time interval dt:

$$\begin{aligned} |\vec{\mathbf{v}}|^2 &= \left|\frac{ds}{dt}\right|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\ |\vec{\mathbf{v}}| = \text{speed} &= \left|\frac{ds}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \end{aligned} \tag{9}$$

The velocity $\vec{\mathbf{v}}$, which is always directed along the curve, can now be nicely expressed in terms of the speed $|\vec{\mathbf{v}}|$ and our tangent vector $\hat{\mathbf{T}}$:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\vec{\mathbf{r}}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\hat{\mathbf{T}} = |\vec{\mathbf{v}}|\hat{\mathbf{T}} \tag{10}$$

This is an important, if perhaps obvious result: we can describe the particle's motion completely if we know its current speed $|\vec{\mathbf{v}}|$ and orientation $\hat{\mathbf{T}}$.

Armed with our new expressions, we can readily calculate the actual distance covered over an arbitrary curve described by $\vec{\mathbf{r}}(t)$. The actual distance covered is calculated by integrating the infinitesimal displacements from some starting time t_i to some ending time t_f:

$$s_f - s_i = \text{path length} = \int \left[\frac{ds}{dt}\right]\,dt = \int_{t_i}^{t_f} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{t_i}^{t_f} \sqrt{v_x^2 + v_y^2}\,dt = \int_{t_i}^{t_f} |\vec{\mathbf{v}}|\,dt \tag{11}$$

This is exactly analogous to what we did for one-dimensional motion – the change in position is found by integrating velocity over a time interval

$$x_f - x_i = \text{distance covered} = \int_{t_i}^{t_f} v_x \, dt = \int_{t_i}^{t_f} \left[\frac{dx}{dt}\right] \, dt \tag{12}$$

For the case of motion along a parametric curve described by $x(t)$ and $y(t)$, we can now readily calculate the instantaneous velocity and speed at any point of the path, and further, we can find the actual distance along the curve between any two points. What if we don't know the curve parametrically, but have an explicit form $y(x)$? In that case, we can use the following substitutions:

$$x = t \qquad \text{and} \qquad y = f(x)$$
$$dx = dt \tag{13}$$
$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} \tag{14}$$

This turns our equation for the path length into something a little nicer looking, giving the length of a curve between two points $x_i = x(t_i)$ and $x_f = x(t_f)$:

$$s_f - s_i = \text{length of curve between } x_i \text{ and } x_f = \int_{x_i}^{x_f} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \tag{15}$$

As an aside, one can also do the same for a curve defined in polar coordinates, $r = f(\theta)$. We will only record the result, for completion:

$$s_f - s_i = \text{length of curve between angles } \alpha \text{ and } \beta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \tag{16}$$

What we have really done, in fact, is turn *any* motion problem along any strange path into an equivalent one-dimensional problem. The price we pay is having to compute what is often a very nasty integral ...

2 Curvature of a path

We can now determine the distance covered along an arbitrary path, which effectively turns any motion problem into a one-dimensional version, but this is not the whole story. For instance, you probably realize that there is a difference between traveling 60 mi/hr along a straight line and taking a sharp curve at 60 mi/hr. Going around the curve, we feel a side-to-side acceleration, which increases as our speed increases and the radius of the curve decreases. Clearly, at a given speed the *curvature* of the path we are one has an influence over the total acceleration experienced. What we require is a reasonably-rigid mathematical description of curvature.

2.1 Defining curvature

Qualitatively, we might compare speed and curvature, by saying that the latter is a measure of how much our direction changes over a given change in position:

speed – change in position with time
curvature – change in direction with position

How do we specify how an object's direction changes with position? We have already come up with a way to specify the direction of travel, our unit vector $\hat{\mathbf{T}}$. This can be used to define curvature, as we will see below. However, it is a bit more illustrative to start simply. We will define the orientation at a particular point on a curve to be the angle φ that a tangent line at that point makes with the x axis, as shown below:

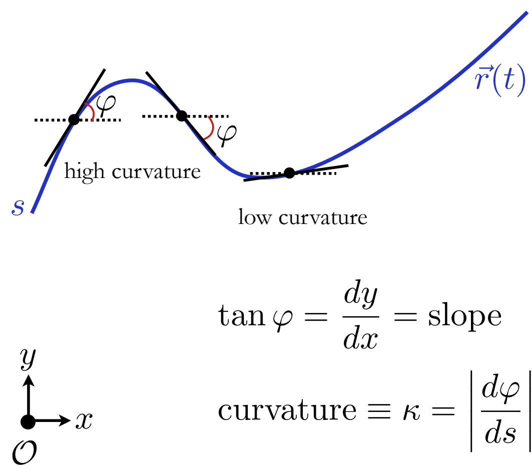

Figure 2: *We can describe the curvature of a path at any point in terms of how quickly the orientation of a tangent line changes with position. Given the angle φ a tangent line makes with the x axis, curvature is $d\varphi/ds$. Since the slope of the curve at any point is $\tan\varphi$, curvature is related to how rapidly the slope of the path changes with position.*

The higher the curvature in a region, the more rapidly that the tangent angle φ changes with position. Based on our definition of curvature above, we want to know how much φ changes for an incremental distance along the curve, or

$$\text{curvature} \equiv \kappa = \left|\frac{d\varphi}{ds}\right| \tag{17}$$

Based on the equation above, curvature must have the units of inverse length. Another convenient quantity is the *radius of curvature*, $R \equiv 1/\kappa$. The radius of curvature at a given point on a curve has a simple physical interpretation: what is the radius of a circle that approximates the curve near that point? Clearly, for a circle, the radius of curvature is just the radius of the circle. The radius of curvature is in essence a way of locally approximating a curve by segments of a circle. The smaller the radius of the circle, the "tighter" the curve is, and the larger the curvature.

Writing Eq. 17 another way, if we move an incremental distance ds along a path, we expect the orientation to change by an amount $|d\varphi| = \kappa|ds|$. We can bet a better feeling for curvature if we remember that $\tan\varphi$ is just the slope of the curve:

$$\tan\varphi = \frac{dy}{dx} = \text{slope} \tag{18}$$

What we are really saying, in a way, is that high curvature corresponds to a region where the slope changes rapidly with position. Fine. We have a *definition* of curvature, but thus far no way to actually calculate it.

2.2 Calculating curvature

Based on our definition of curvature in Eq. 17, if we apply the chain rule[i] we can make a little progress:

$$\frac{d\varphi}{ds} = \frac{d\varphi}{dx}\frac{dx}{ds} = \frac{d\varphi/dx}{ds/dx} \tag{19}$$

The denominator, ds/dx, we already know – it is trivially found by differentiating Eq. 15:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{20}$$

That leaves only the numerator, $d\varphi/dx$. With the benefit of hindsight, we can use the one relationship we already have involving both φ and x, Eq. 18, and differentiate it with respect to x:

$$\begin{aligned}
\tan\varphi &= \frac{dy}{dx} \\
\frac{d}{dx}\left(\tan\varphi\right) &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
\left(\sec^2\varphi\right)\frac{d\varphi}{dx} &= \frac{d^2y}{dx^2} \\
\frac{d\varphi}{dx}\left(1+\tan^2\varphi\right) &= \frac{d^2y}{dx^2} \\
\frac{d\varphi}{dx} &= \frac{d^2y}{dx^2}\frac{1}{1+\tan^2\varphi} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} \qquad (21)
\end{aligned}$$

Combining this with Eqs. 17, 19, and 20, we can calculate the curvature from the explicit equation for a path $y(x)$ and its derivatives:

$$\kappa = \left|\frac{d\varphi}{ds}\right| = \left|\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}\;\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right| = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{22}$$

One can also develop an equivalent expression for curvature based on the parametric expression for the same path, $y(t)$ and $x(t)$.

$$\kappa = \frac{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right|}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}} \tag{23}$$

This will be easier to derive[ii] once we've done a few other things. It is a nice exercise to convince yourself

[i] We also need to assume that dx/ds exists, $ds/dx \neq 0$, and $ds/dx = 1/(dx/ds)$. For the sort of well-defined functions we deal with, representing physical quantities, this does not represent a problem. We will be a little slapdash with some points like this, knowing that our mathematician colleagues have worked out the formalities already.

[ii] An example derivation similar to our method above can be found at `http://mathworld.wolfram.com/Curvature.html`

that it is true, however ☺

3 The normal unit vector

So far, for an arbitrary path – parametrically or explicitly defined – we can now define and calculate the path length, speed, and curvature. Further, we have a handy unit vector $\hat{\mathbf{T}}$ that tells us at any point on the path, for any instant in time, what the direction of travel is. What else do we need to fully describe motion along this path? Only one more thing: a corresponding unit vector that tells us in which direction we are turning. It is not enough to know the degree of curvature, we need to know the orientation as well!

Fortunately, there is a simple way to describe the direction we are turning for a given path. Recall $\hat{\mathbf{T}}$ is a unit vector which is always tangent to the object's path, defined by

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \frac{dx}{ds}\,\hat{\boldsymbol{\imath}} + \frac{dy}{ds}\,\hat{\boldsymbol{\jmath}}$$

An equivalent definition of $\hat{\mathbf{T}}$ results from simply saying that we want a unit vector pointing along the current velocity. That is, take the current velocity vector, and divide by its magnitude to come up with a unit vector:

$$\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \tag{24}$$

If we want to know the direction we are turning, it is sufficient to find a unit vector $\hat{\mathbf{N}}$ which is always *normal* to $\hat{\mathbf{T}}$. This gives two possible directions, and we pick the one that describes the direction the path is curving toward, rather than the direction the path is curving away from. This is probably easier to grasp graphically:

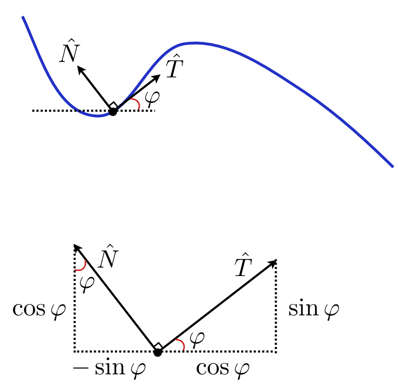

Figure 3: Upper: *at any point on the curve, we can define a unit vector* $\hat{\mathbf{T}}$ *that points tangentially along the curve in the direction of travel. We now define a unit vector* $\hat{\mathbf{N}}$ *which is perpendicular to* $\hat{\mathbf{T}}$ *and points in the direction the path is turning.* **Lower:** *Given our expression for* $\hat{\mathbf{T}}$*, simple geometry lets us determine* $\hat{\mathbf{N}}$*.*

Thus, given $\hat{\mathbf{T}}$ and a particular curve, $\hat{\mathbf{N}}$ is strictly defined. For a circular path, for instance, $\hat{\mathbf{N}}$ would always point toward the center of the circle, while for a parabolic path resulting from free-fall motion, $\hat{\mathbf{N}}$ always points toward or parallel to the ground.

From the geometry in the lower portion of Fig. 3, we can already figure out the component form of $\hat{\mathbf{N}}$. It must be

$$\hat{\mathbf{N}} = -\sin\varphi\,\hat{\boldsymbol{\imath}} + \cos\varphi\,\hat{\boldsymbol{\jmath}} \tag{25}$$

It is easy to verify that $|\hat{\mathbf{N}}|=1$, and that $\hat{\mathbf{T}}\cdot\hat{\mathbf{N}}=0$ (meaning that $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are always perpendicular). An equivalent definition of $\hat{\mathbf{N}}$ is based on our earlier notion that $\hat{\mathbf{N}}$ should tell us the direction we are turning. More precisely, this means that $\hat{\mathbf{N}}$ should represent the change in $\hat{\mathbf{T}}$ as one goes an incremental distance ds along the path. Since $\hat{\mathbf{T}}$ itself tells us the direction of the instantaneous velocity, the change in $\hat{\mathbf{T}}$ with ds tells us how rapidly the *direction* of the velocity changes along the curve. Mathematically, this is just

$$\hat{\mathbf{N}} = \frac{\dfrac{d\hat{\mathbf{T}}}{ds}}{\left|\dfrac{d\hat{\mathbf{T}}}{ds}\right|} \tag{26}$$

Here of course we remembered to divide $d\hat{\mathbf{T}}/ds$ by its magnitude to ensure that we created a unit vector.

4 Acceleration along a curved path

Now we have everything we need to describe motion along a perfectly arbitrary path.[iii] We can, for a given trajectory, find the path length, velocity, and curvature. At a given point, we can precisely determine the direction of travel, the turning direction, and even how rapidly the turn is being executed. So what![iv]

The Big Deal here is that now we can turn motion along *any* path into an equivalent one-dimensional problem. That is, all motion problems are one-dimensional. How? Let us simply calculate the velocity and acceleration for an arbitrary path $\vec{\mathbf{r}}(t)$, and it will be apparent. Given a path $\vec{\mathbf{r}}(t)$, we already know the velocity from Eq. 10:

$$\begin{aligned} \vec{\mathbf{r}}(t) &= x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} \\ \vec{\mathbf{v}}(t) &= \frac{d\vec{\mathbf{r}}}{dt} = \frac{ds}{dt}\,\hat{\mathbf{T}} \end{aligned} \tag{27}$$

Using our new machinery, the velocity's magnitude is just the speed, and its direction is $\hat{\mathbf{T}}$. How about the acceleration? We just differentiate the velocity with respect to time, and apply the chain rule:

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\,\hat{\mathbf{T}}\right) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{ds}{dt}\frac{d\hat{\mathbf{T}}}{dt} = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{ds}{dt}\frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt} \tag{28}$$

Now we have to be careful that $\hat{\mathbf{T}}$ is a *local* quantity, which changes along our path. It is neither constant in time, nor it is constant along the path s. Using the cartesian expression for $\hat{\mathbf{T}}$, Eq. 6, it is easy to calculate

[iii] With the caveats that the path must be continuous, its spatial derivatives must be continuous, *etc.*

[iv] We have also confined ourselves to two dimensions. It is not a big trick to extend these ideas to three dimensions. You will probably do this in a future calculus class.

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{d}{ds}\left(\cos\varphi\,\hat{\boldsymbol{\imath}} + \sin\varphi\,\hat{\boldsymbol{\jmath}}\right) = -\frac{d\varphi}{ds}\sin\varphi\,\hat{\boldsymbol{\imath}} + \frac{d\varphi}{ds}\cos\varphi\,\hat{\boldsymbol{\jmath}}$$
$$= \frac{d\varphi}{ds}\left(-\sin\varphi\,\hat{\boldsymbol{\imath}} + \cos\varphi\,\hat{\boldsymbol{\jmath}}\right) = \frac{d\varphi}{ds}\,\hat{\mathbf{N}} \tag{29}$$

Putting it all together,

$$\vec{\mathbf{a}}(t) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2\frac{d\varphi}{ds}\,\hat{\mathbf{N}} \tag{30}$$

Now we can make two important identifications: first, ds/dt is nothing more than the speed along the path, $|\vec{\mathbf{v}}|$; second, the quantity $d\varphi/ds$ is just the curvature κ. Finally,

$$\vec{\mathbf{a}}(t) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}} = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{|\vec{\mathbf{v}}|^2}{R}\,\hat{\mathbf{N}} \equiv a_N\,\hat{\mathbf{T}} + a_T\,\hat{\mathbf{N}} \tag{31}$$

Note that the second form uses the radius of curvature $R \equiv 1/\kappa$. There are two terms to the acceleration: one parallel to the path (a_T), and one perpendicular to the path (a_N). The first is the acceleration along the path (the $\hat{\mathbf{T}}$ direction), and it depends only on the change in speed with time, d^2/dt^2. This is nothing new – in one dimension, the acceleration is precisely the same. Given a path $x(t)$, the acceleration is d^2x/dt^2 along the x direction.

The second term is new, however. It represents an acceleration *perpendicular* to the path (the $\hat{\mathbf{N}}$ direction), or a side-to-side acceleration as one is going around a curve. This is known as *centripetal acceleration*, acceleration which has nothing to do with changing speed, but results only from changing direction. The larger the curvature of the path (or the smaller the radius of curvature, the "tighter" the turn), and the larger the speed along the path, the larger the side-to-side acceleration is. This is quite familiar. Taking a sharp turn at 60 mi/h is a very different experience than taking the same turn at 30 mi/h, and at a given speed, a tighter corner certainly gives a larger side-to-side acceleration. Later, when we associate acceleration with *force*, this will take on new meaning.

Just so it sinks in, let us reemphasize what this second term means: the faster you take a given curve, the larger the side-to-side force, and the tighter the curve at a given speed, the larger the side-to-side force. Acceleration can result from velocity *changing direction only*, not just velocity changing magnitude. If the magnitude of velocity is the same (constant speed), but its direction changes rapidly, there is a large side-to-side acceleration but none along the path. If the magnitude of velocity changes rapidly, but the path is straight, there is a large acceleration along the path, but none side-to-side.

So what have we really done? We have found that motion along a curved path can be described just like one-dimensional motion, provided we define a *local* coordinate system defined by the unit vectors $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$. Rather than defining a global $x - y$ coordinate system, we change our coordinate system at every instant to align with the path we are on. This local description of geometry and motion is something that you will encounter more and more in future physics courses. At any instant in time, however, the local unit vectors $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are just as good as $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ – they are orthogonal (perpendicular), and the acceleration components behave just like you would expect:

$$|\vec{\mathbf{a}}|^2 = a_T^2 + a_N^2 \qquad \text{and} \qquad \theta_T = \tan^{-1}\left(\frac{a_N}{a_T}\right) \tag{32}$$

5 A few loose ends

We can quickly derive a few other interesting relationships. First, consider $\vec{\mathbf{v}}\times\vec{\mathbf{a}}$. Using our general expression for $\vec{\mathbf{a}}$ from Eq. 31,

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \left(\frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}}\right) \tag{33}$$

The vector product $\times$ is distributive, and scalar multiplication is communtative, so this is really

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \frac{d^2s}{dt^2}\,\vec{\mathbf{v}} \times \hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\vec{\mathbf{v}} \times \hat{\mathbf{N}} \tag{34}$$

By definition, $\hat{\mathbf{T}}$ points along the direction of $\vec{\mathbf{v}}$, so the two vectors are parallel. Therefore, their vector product is zero. Thus,

$$\vec{\mathbf{v}} \times \vec{\mathbf{a}} = \kappa|\vec{\mathbf{v}}|^2\,\vec{\mathbf{v}} \times \hat{\mathbf{N}} \tag{35}$$

The angle between $\vec{\mathbf{v}}$ and $\hat{\mathbf{N}}$, by construction, is $90°$ – they are always perpendicular. Worrying only about magnitudes, their cross product is then $|\vec{\mathbf{v}}||\hat{\mathbf{N}}|=|\vec{\mathbf{v}}|$, since $\sin 90°=1$ and $|\hat{\mathbf{N}}|=1$. Thus,

$$|\vec{\mathbf{v}} \times \vec{\mathbf{a}}| = \kappa|\vec{\mathbf{v}}|^3 \tag{36}$$

This gives us an alternative, coordinate-free expression for the curvature κ in terms of $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ alone:

$$\kappa = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3} \tag{37}$$

6 Does it make sense? The special case of straight-line and free-fall motion

Coming soon ...

7 Does it make sense? The special case of circular motion

Does our expression for curvature make any sense when applied to a familiar path? Let us consider the special case of motion along a circular path.

Coming soon: applying the general equations to the special case of circular motion ...

A Curiosities

A.1 Analogy between local coordinates and rotation matricies

It is interesting (and sensible, if you dwell on it) that the normal and tangential unit vectors bear a striking relationship to a rotation of coordinate systems. Take a normal (x, y) system, and pick a point $P(x, y)$. After a *counterclockwise* rotation of θ about the origin, the new coordinates after rotation $P'(x, y)$ are

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Compare this to the unit vectors $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ written in matrix form:

$$\begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \end{bmatrix} \tag{38}$$

Redefining a local coordinate axis in terms of tangential and normal unit vectors is precisely the same as rotating an existing $x - y$ system to align with a particular curve at a particular point. The main difference in our approach is that *the tangential and normal vectors are changing in time, and only locally defined.* Rather than rotating coordinates just once to change to a new frame, we do this after every infinitesimal time increment dt.

There is no need for a *global* $x - y$ coordinate system, so long as we can always define $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ locally. As it turns out, the deepest laws of physics we know (such as general relativity) need to be described in terms of local or *differential* geometry, similar to what we have done above. Nature has no preferred coordinate system or origin, and it is the local description of geometry and space we introduce here that is required.

You will encounter these ideas again in a future physics course. For the rest of the semester, however, this is just something we'll let you ponder ☺