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A Crash Course in Fluid Dynamics

Contents

In your introductory mechanics class, you have no doubt dealt with hydrostatics, and the forces and energy of fluids which are at rest. The purpose of these notes is to introduce you to the problem of a moving fluid, and derive the equations relating the pressure and velocity distribution within a fluid. As a special case, we will derive Bernoulli's equation for incompressible, viscosity-free fluids, and as a more general case we will derive the Navier-Stokes equation for incompressible fluids. We will then apply these results to a (relatively) simple case, the flow of low-speed air past a dense sphere.

1 The Continuity Equation

The starting point[i] for our treatment of fluids will be the derivation of the *continuity equation* for a fluid. A continuity equation, if you haven't heard the term, is nothing more than an equation that expresses a conservation law. In the case of continuous media, such as a fluid, our conservation law is conservation of matter. In electromagnetism, one continuity equation expresses conservation of charge.

Qualitatively, a generic continuity equation for mass reads something like this:

$$\text{(rate of mass accumulation)} + \text{(rate of mass out)} - \text{(rate of mass in)} = 0 \tag{1}$$

If you replace "mass" with "charge" or "momentum" you can imagine all sorts of continuity equations that fall under the general heading of "conservation of stuff." We can be a bit more precise by applying our continuity equation to a specific volume of space V, which is defined by a bounding surface S. In this case, the net rate at which mass accumulates inside V depends on the net rate at which mass passes through S, either coming in or going out:

$$\text{(rate of mass accumulation in } V) + \text{(net rate of mass crossing } S) = 0 \tag{2}$$

This is basically just bookkeeping. If the amount of mass in V is static, then it must be true that the amount of matter entering through S is the same as the amount of matter leaving through S. If the amount of mass in V is increasing, then there must be a net flow of matter in through S. Since we wish to deal with continuous substances like fluids, rather than the individual particles we usually deal with in mechanics, it is most convenient to put our equations in terms of the density of the substance ρ.

Consider a tiny cube of our substance of dimensions $\Delta x\,\Delta y\,\Delta z$. The mass of this cube is simply $\rho\,\Delta x\,\Delta y\,\Delta z$. If we have a net flow of our substance through this cube, let's say in the x direction, how does the mass of the cube change with time? If the substance is incompressible, and the cube

[i]Much of this document is based on Ch. 2 of D.R. Poirier and G.H. Geiger, *Transport Phenomena in Materials Processing*, TMS, Warrendale, PA, 1994) and Ch. 40-41 of the Feynman Lectures on Physics, vol. II

remains completely full, then the mass *doesn't* change, of course. However, in the general case, we just need to keep track of how much mass is in the cube at any moment, and how much mass enters and leaves.

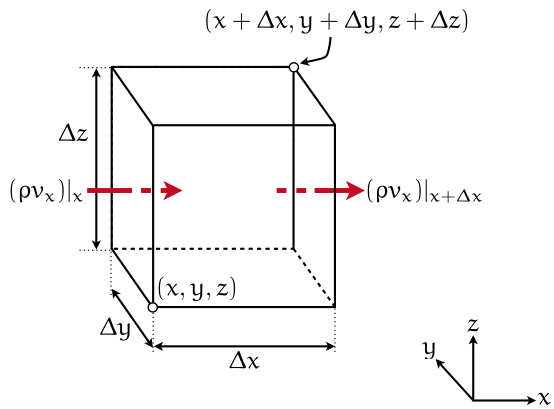


Figure 1: *Volume element fixed in space with fluid flowing through it.*

We will presume that our cube is nicely aligned along the x, y, and z axes, and that there is a net flow of our substance with velocity $\mathbf{v}$, as shown in Fig. 1. We will assume that the density of our substance is constant. If we look first at the faces of the cube perpendicular to the x axis (i.e., the faces whose area normals are parallel to the x axis), the net flow through the cube along the x axis can be found be comparing the rate at which mass enters one side and leaves the other. The rate of mass flowing through the left side face at x is

$$\left.\frac{\partial m}{\partial t}\right|_x = \left.\frac{\partial}{\partial t}\left(\rho V\right)\right|_x = \left.\frac{\partial}{\partial t}\left(\rho \Delta x \Delta y \Delta z\right)\right|_x = \left.\Delta y \Delta z \left(\rho v_x\right)\right|_x \tag{3}$$

This is just the familiar result that the mass flow rate through a pipe is product of the velocity of the flow, the fluid density, and the pipe's cross-sectional area. In the same manner, we can find the flow rate through the right side face at $x + \Delta x$,

$$\left.\frac{\partial m}{\partial t}\right|_{x+\Delta x} = \left.\Delta y \Delta z \left(\rho v_x\right)\right|_{x+\Delta x} \tag{4}$$

We can proceed similarly for the other two pairs of faces perpendicular to the y and z axes, and then add up all the terms for fluid entering or leaving the cube to come up with a mass balance. If, when we add up the rates for all the sides, we have a non-zero result, then we must be either accumulating mass inside our cube, or it is experiencing a net loss in mass. Either way, the *accumulation* in mass inside our cube of constant volume can only reflect a change in density, and since we consider a constant volume $\Delta V = \Delta x\, \Delta y\, \Delta z$,

$$(\text{mass accumulation}) = \frac{\partial m}{\partial t} = \frac{\partial(\rho V)}{\partial t} = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} \tag{5}$$

Our mass balance is then simply relating this rate of mass accumulation to the net flow through the cube:

$$\begin{aligned}\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \left(\rho v_x \Big|_x - \rho v_x \Big|_{x+\Delta x} \right) + \Delta x \Delta z \left(\rho v_y \Big|_y - \rho v_y \Big|_{y+\Delta y} \right) \\ + \Delta x \Delta y \left(\rho v_z \Big|_z - \rho v_z \Big|_{z+\Delta z} \right)\end{aligned} \tag{6}$$

Next, we can divide by $\Delta x \Delta y \Delta z$ and take the limit of infinitesimal dimensions. Recalling the definition of the derivative, we arrive at the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x} \rho v_x + \frac{\partial}{\partial y} \rho v_y + \frac{\partial}{\partial z} \rho v_z \right) = -\nabla \cdot (\boldsymbol{\rho} \mathbf{v}) \qquad \text{(general continuity equation)} \tag{7}$$

In many situations, such as the flow of air at very low velocities or the flow of water in general, we may assume to a good approximation that the fluid has approximately constant density (i.e., it is incompressible). If that is the case, then $\nabla \cdot (\boldsymbol{\rho} \mathbf{v}) = \rho \nabla \cdot \mathbf{v}$, and $\partial \rho / \partial t = 0$ since if the fluid's density is constant it can't vary with time. This leads us to:

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{(continuity equation, incompressible fluid)} \tag{8}$$

As a comparison, the equivalent continuity equation in electromagnetism is conservation of charge, which you might have seen:

$$\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0 \tag{9}$$

where ρ is charge density and $\mathbf{j}$ current density. In this case, the continuity equation states that the charge density in a region can only change if there is a net flow of charge (a current) into or out of that region. The analog of an incompressible fluid in electromagnetism is *electrostatics*, or no net motion of charge, which means the continuity equation is simple $\nabla \cdot \mathbf{j} = 0$. Pushing the analogy a bit further, the analog of electric current density $\mathbf{j}$ in a fluid is a mass current $\rho \mathbf{v}$, the net transport of mass through a unit area.

For the most part, in this document **we will assume that the fluid density is constant, and treat only incompressible fluids.** This restriction is reasonable for a myriad of practical situations, but it will not allow us to consider, e.g., density waves such as sound propagation. How good is the approximation? Compressibility is defined as

$$\beta = -\frac{1}{V}\frac{\partial V}{\partial P} \tag{10}$$

That implies $\delta V/V \sim \beta\delta P$. For water at 0°C, $\beta \approx 5 \times 10^{-10}\,\mathrm{Pa}^{-1}$, which means that for a 1% change in relative volume we require a pressure of 5×10^{7} Pa, or about 500 atm. For most practical purposes, we can therefore consider water and similar fluids to be incompressible. [ii]

1.1 The Continuity equation in spherical coordinates

Using the vector form of the continuity equation, we can reformulate it for different coordinate systems relevant to specific problems by expanding the divergence operator $\nabla\cdot$ appropriately. In spherical coordinates, we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 \partial v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\rho v_\varphi\right) = 0 \tag{11}$$

Our main problems of interest are the slow flow of a fluid past a dense sphere, and the flow through a pipe. If the fluid flow is along the z axis, the problem is symmetric about the z axis, and we may neglect the φ components of velocity. In other words, the problem is essentially two-dimensional, thanks to the rotational symmetry about the z axis. In this special case,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 \partial v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho v_\theta \sin\theta\right) = 0 \tag{12}$$

2 Static fluids

Next, we will need the equation for the forces and momentum in the fluid. Let us consider a completely static volume of fluid, with no net flow in any direction. If we know the pressure at some point within the fluid (say, at its bottom surface) is P_o, then at any point a height h above that level, the pressure is just $P = P_o - \rho g h$ where g is the gravitational acceleration, and ρ the fluid density (Fig. 2). Put another way, the pressure as a depth h differs from our reference level only by the weight of the fluid in a column of height h.

We can turn this equation around: if P_o is just an arbitrary, constant reference pressure, then this also implies that *anywhere* in the fluid $P + \rho g h = P_o$ must be constant! Actually, this is not so surprising either. If we multiply everything by a volume of interest, we are merely stating

[ii] One can also show that $\beta \approx \frac{1}{\rho c^2}$, where ρ is the fluid density and c is the speed of sound. Given $c \approx 1480\,\mathrm{m/s}$ in water, we arrive at $\beta \approx 5 \times 10^{-10}\,\mathrm{Pa}^{-1}$. See https://en.wikipedia.org/wiki/Compressibility

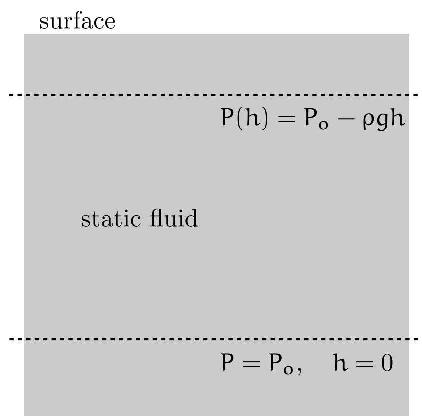


Figure 2: *Pressure variation with depth in a static fluid. The pressure at a height h above a reference level is smaller by the weight per unit area of the fluid above the reference level.*

conservation of energy.

$$P_oV = PV + \rho Vgh = PV + mgh \tag{13}$$

The work done in increasing the pressure on a given constant volume is $P(V - V_o)$, and this work must be accounted for by the change in gravitational potential energy, mgh. Again, in dealing with continuous matter such as a fluid, it is more convenient to recast all of our equations in terms of *density* rather than mass and volume. In this light, gh is just the gravitational potential *per unit mass*, so what we are really saying is that *pressure plus gravitational potential is a constant* for a static fluid, or that pressure itself is a sort of volumetric potential. Thus, if we define a gravitational potential per unit mass $\phi = gh$, we have

$$P + \rho\phi = \text{const} \qquad \text{with} \qquad \phi = U_{\text{grav}}/m = gh \tag{14}$$

Now we have an energy balance for our static fluid, it is only a bit of mathematics to find a force balance. If we consider a one-dimensional fluid, we know that force is just the spatial derivative of the potential energy, $F_x = -dU/dx$. The same will hold true of the potential energy per unit volume, which amounts to taking the spatial derivative of both sides of Eq. 14. In one dimension, this gives

$$\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}(\rho\phi) = 0 \tag{15}$$

The force has two terms: the first tells us that fluids move in response to a pressure gradient, from high to low, and the second tells us that fluids flow in response to gravitational force, downhill. If we consider only fluids of constant density (incompressible fluids), this simplifies to

$$\frac{\partial P}{\partial x} + \rho\frac{\partial \phi}{\partial x} = 0 \tag{16}$$

In three dimensions, we need only replace the spatial derivative with a gradient:

$$\nabla P + \nabla(\rho\phi) = 0 \qquad \text{(compressible)} \tag{17}$$

$$\nabla P + \rho\nabla\phi = 0 \qquad \text{(incompressible)} \tag{18}$$

This is nothing more than a Newton's law force balance for our stationary fluid, if we recognize that $\rho\nabla\phi$ is the force (per unit volume): in static equilibrium, the force per unit volume is precisely balanced by a gradient in pressure.

This equation is the complete description of hydrostatics, though it is quite a bit more complicated than it looks: there is no general solution. If the density of the fluid varies spatially ($\nabla\rho \neq 0$ somewhere), our continuity equation above tells us that there is no way that a *static* equilibrium can be maintained, we must have also have time-varying density. [iii] Only if ρ is constant in space do we have a simple solution for hydrostatics, viz., $P + \rho\varphi = \text{const}$.

3 Moving fluids: Equations of motion without viscosity ("Dry Water")

What to do if the fluid is not static? We already know the continuity equation in general, but we still need to consider a more general force balance for our fluid. What we have derived above is the equilibrium condition for a static fluid, generalizing just means letting the pressure and potential gradient terms become unbalanced to yield a net acceleration. In the absence of viscous forces, this would simply be

$$\rho \times (\text{acceleration}) = -\nabla P - \rho\nabla\phi \tag{19}$$

The left side is the net force per unit volume, and the first two terms on the right are our pressure and potential gradients. Already, if these terms on the right are unbalanced (e.g., if we have a spatially-varying density) we will have a net acceleration of the fluid, and hence motion. What does the acceleration term look like?

What we really need to find is $\Delta\mathbf{v}/\Delta t$ for infinitesimal Δt, that is our acceleration. Just from the mathematics of partial derivatives, we can say quite a lot already. Say we know the velocity of a infinitesimal volume of fluid at some particular point in space and time, $\mathbf{v}(x, y, z, t)$. What is the velocity of the *same* bit of fluid at some later time $t + \Delta t$ when the bit of fluid is at a neighboring point $(x + \Delta x, y + \Delta y, z + \Delta z)$? From the definition of partial derivatives, for small changes in x,

[iii] Strictly, for a fluid of constant density, a spatially-varying density in Eq. 7 implies that the velocity field must have zero divergence, or be zero everywhere to have a density which does not vary in time. Only the case $v = 0$ corresponds to a truly static situation, and thus, if ρ has any spatial variation, a time variation is implied.

y, z, and t (i.e., to first order) we can write the change in velocity as

$$\Delta\mathbf{v} = \mathbf{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - \mathbf{v}(x,y,z,t) \tag{20}$$

$$\approx \frac{\partial\mathbf{v}}{\partial x}\Delta x + \frac{\partial\mathbf{v}}{\partial y}\Delta y + \frac{\partial\mathbf{v}}{\partial z}\Delta z + \frac{\partial\mathbf{v}}{\partial t}\Delta t \tag{21}$$

This is not incredibly useful, as such, but we can multiply and divide every spatial derivative by Δt to put this in a more interesting form:

$$\Delta\mathbf{v} = \frac{\partial\mathbf{v}}{\partial x}\Delta x + \frac{\partial\mathbf{v}}{\partial y}\Delta y + \frac{\partial\mathbf{v}}{\partial z}\Delta z + \frac{\partial\mathbf{v}}{\partial t}\Delta t \tag{22}$$

$$= \frac{\partial\mathbf{v}}{\partial x}\frac{\Delta x}{\Delta t}\Delta t + \frac{\partial\mathbf{v}}{\partial y}\frac{\Delta y}{\Delta t}\Delta t + \frac{\partial\mathbf{v}}{\partial z}\frac{\Delta z}{\Delta t}\Delta t + \frac{\partial\mathbf{v}}{\partial t}\Delta t \tag{23}$$

$$= \frac{\partial\mathbf{v}}{\partial x}v_x\Delta t + \frac{\partial\mathbf{v}}{\partial y}v_y\Delta t + \frac{\partial\mathbf{v}}{\partial z}v_z\Delta t + \frac{\partial\mathbf{v}}{\partial t}\Delta t \tag{24}$$

$$= \left(\frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z + \frac{\partial\mathbf{v}}{\partial t}\right)\Delta t \tag{25}$$

The acceleration, $\Delta\mathbf{v}/\Delta t$ in the limit $\Delta t\to 0$, is then

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z + \frac{\partial\mathbf{v}}{\partial t} \tag{26}$$

This might not look like much, but if we look and rearrange it carefully we can recognize a nicer vector form:

$$\frac{d\mathbf{v}}{dt} = v_x\frac{\partial\mathbf{v}}{\partial x} + v_y\frac{\partial\mathbf{v}}{\partial y} + v_z\frac{\partial\mathbf{v}}{\partial z} + \frac{\partial\mathbf{v}}{\partial t} \tag{27}$$

$$= \left[(v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}})\cdot\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\right]\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} = (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} \tag{28}$$

Can you see why it must be $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ and not, e.g., $\mathbf{v}\cdot(\nabla\mathbf{v})$? (If for no other reason, the former is a vector while the latter is a scalar!)

Having found the acceleration, in the absence of viscous forces our equation of motion is complete:

$$\frac{d\mathbf{v}}{dt} = \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \rho\frac{\partial\mathbf{v}}{\partial t} = -\nabla P - \rho\nabla\phi \qquad \text{(equation of motion, no viscosity)} \tag{29}$$

3.0.1 Rotation

We can add a bit more physical content to our equation of motion by defining a new field from the curl of the velocity, $\mathbf{\Omega}=\nabla\times\mathbf{v}$. This quantity is called the *vorticity* of the fluid, and it characterizes the circulation of the fluid. If $\mathbf{\Omega}=0$ everywhere, the fluid is said to be *irrotational*. By introducing the vorticity, we can separate the terms in our equation of motion to characterize two basic cases: fluids that swirl, and those that do not. (It might help to recall the fundamental theorem of vector calculus, which roughly states that we can build any reasonable vector field out of the sum of an irrotational (zero curl) field and a solenoidal (zero divergence) field.)

If we are only interested in fluids that do not circulate, this will allow considerable simplification. In order to achieve this separation, we can also make use of the following vector identity to introduce terms that contain $\nabla \times \mathbf{v}$:[iv]

$$(\mathbf{v} \cdot \nabla)\,\mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2}\nabla\,(\mathbf{v} \cdot \mathbf{v}) = \mathbf{\Omega} \times \mathbf{v} + \frac{1}{2}\nabla v^2 \tag{30}$$

This allows us to put our equation of motion in the following form:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{\Omega} \times \mathbf{v} + \frac{1}{2}\rho\nabla v^2 = -\nabla P - \rho\nabla\phi \tag{31}$$

The physical content of the vorticity field is perhaps more apparent if we we recall the fundamental theorem for curls, which states that integrating the curl of a function over a surface S is equivalent to taking a line integral of that function over a curve C bounding the surface:

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_S \mathbf{\Omega} \cdot d\mathbf{a} = \oint_C \mathbf{v} \cdot d\mathbf{l} \tag{32}$$

The line integral of the velocity around a closed loop is nothing more than the net circulation of the fluid, so what this tells us is that the vorticity $\mathbf{\Omega}$ is just the net circulation of the fluid around an infinitesimal closed loop. Consider the case where we have *pure* rotational motion of a fluid, such as a perfect circular flow of fluid inside a bucket. At a given radius r from the center of rotation, this gives $2\pi r v = \pi r^2 \Omega$, or $\omega = \Omega/2$. The angular velocity of the fluid (or a small particle placed in the fluid) at any given radius is just half the vorticity.

We can go still further with vorticity. If we are only interested in the velocity field in the fluid, we can eliminate pressure from Eq. 31. If we take the curl of both sides of Eq. 31, and remember that $\nabla \times (\nabla f) = 0$ for any f, we have

$$\rho\nabla \times \frac{\partial \mathbf{v}}{\partial t} + \rho\nabla \times (\mathbf{\Omega} \times \mathbf{v}) = 0 \tag{33}$$

$$\text{or} \qquad \frac{\partial \mathbf{\Omega}}{\partial t} + \nabla \times (\mathbf{\Omega} \times \mathbf{v}) = 0 \tag{34}$$

Along with the definition of vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$ and our continuity equation $\nabla \cdot \mathbf{v} = 0$, this equation is sufficient to find the velocity field of our fluid. From the form of these equations, if we know $\mathbf{\Omega}$

[iv] See https://en.wikipedia.org/wiki/Vector_calculus_identities, and look for "Vector dot product" and the "special case" in particular.

at one particular time, that means we also know both the curl and divergence of $\mathbf{v}$, which means knowledge of $\mathbf{\Omega}$ alone determines v. In fact, there is an even more striking consequence: if we have $\mathbf{\Omega}=0$ everywhere at some instant in time, then $\partial\mathbf{\Omega}/\partial t=0$ as well. If the fluid is irrotational at any time, it is irrotational at *all* times! As nice as this new equation is, we should not forget that we have thrown away all information about the pressure. We would still have to take our velocity field and use Eq. 31 to deduce anything about the pressure.

As an aside, our equations in terms of vorticity have an interesting analogy with magnetism, where we have

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_o \mathbf{j} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{35}$$

Thus, mathematically speaking, velocity is analogous to magnetic field, and vorticity is analogous to current density. Knowledge of the current density throughout space allows us to determine the magnetic field, just as knowledge of the vorticity allows us to determine the velocity.

3.0.2 Irrotational Fluids

Now, taking advantage of this new form, we can consider only irrotational fluids for which $\mathbf{\Omega}=0$, in which case we have the simpler result

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\rho\nabla v^2 = -\nabla P - \rho\nabla\phi \qquad \text{(equation of motion, no viscosity, irrotational)} \tag{36}$$

At this point, we can make another analogy with electromagnetism. The conditions of zero rotation and continuity actually give us enough to solve for the velocity field by themselves:

$$\nabla \cdot \mathbf{v} = 0 \qquad \nabla \times \mathbf{v} = 0 \tag{37}$$

This is just like Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ in free space. This is handy: for an irrotational, incompressible fluid the boundary value problems are often the same as ones we have already solved. What's more, these equations are *linear* differential equations, unlike our more general expressions for fluid flow. This is *only* true for incompressible fluids in irrotational flow. Since the governing equations are linear, that means that the solutions obey superposition: if we have two solutions to the equations, then their sum (or difference) is also a solution, just like in electromagnetism.

3.0.3 Steady flows and Bernoulli's equation

Finally, there is one more simplification we can make for many reasonable cases: the assumption of *steady flow*. This doesn't mean we have nothing happening, it is merely the condition that we have motion of the fluid at constant velocity, $\partial\mathbf{v}/\partial t = 0$. In this case,

$$\frac{1}{2}\rho\nabla v^2 = -\nabla P - \rho\nabla\phi \qquad \text{(equation of motion, no viscosity, irrotational, steady flow)} \quad (38)$$

Since every term in this equation involves a gradient, we may simply integrate both sides to get rid of a gradient from every term, and once we remember to add in an integration constant, we have

$$\frac{1}{2}\rho v^2 + P + \rho\phi = (\text{const}) \quad (39)$$

If we multiply through by a volume of interest, we recover something recognizable:

$$\frac{1}{2}mv^2 + PV + mgh = (\text{const}) \quad (40)$$

This is Bernoulli's theorem, which is just a statement of conservation of energy for an irrotational fluid. Compare this to our starting point for a static fluid, $P + \rho\phi = (\text{const.})$, and you will see that the new term $\frac{1}{2}\rho v^2$ is nothing more than the kinetic energy of the moving fluid!

3.0.4 Example: water draining from a tank

As a quick practical example of what we can do with this, let's consider a case you have all dealt with: water draining in the bathtub. We'll imagine that we have a cylindrical bathtub of radius R, with a drain plug at the bottom in the exact center of the tub. Now we'll fill up the tub, stir it up to get it circulating, and pull the drain plug. We know that at first the water will form a nice spiral circulating down the drain, but the rotation will die out due to viscosity after a short time.[v] After the flow becomes irrotational due to viscous forces, what remains is still a nice conical shape, however. But what is the shape?

Within a given vertical plane, we can calculate the net circulation at a radial distance r from the center as in Eq. 32, the net circulation is $\oint \mathbf{v} \cdot d\mathbf{r} = 2\pi r v_\theta$, where v_θ is the tangential velocity. Once we are in the regime of irrotational flow, however, the circulation can't depend on the radial distance.[vi] Thus, we must have $2\pi r v_\theta = \text{constant}$, and this can only be true if the tangential velocity is proportional to $1/r$. If we express the continuity equation $\nabla \cdot \mathbf{v} = 0$ in cylindrical coordinates, we can find the radial component of the velocity:

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \qquad \Longrightarrow \qquad r v_r = (\text{constant}) \quad (41)$$

[v] Why don't you have to stir the tub to see this effect? It has nothing to do with what hemisphere you're in or the Coriolis effect, it is simply a result of the pipes leading away from the tub having turns in them.

[vi] You might think that the circulation has to be *zero*, but that is not quite true because we include the origin in our surface. The curl of the velocity is zero everywhere *except* the origin, and this gives a constant contribution to the integral of $\nabla \times \mathbf{v}$ over a surface including the origin.

Since v_θ is independent of θ, $\partial v_\theta/\partial\theta = 0$, it must be the case that $\frac{\partial}{\partial r}(r v_r) = 0$, and therefore $r v_r$ must be constant, and it is correct that the radial velocity is proportional to $1/r$. At the air-water boundary, we know that the pressure is simply atmospheric pressure, a constant. Since our fluid is irrotational, incompressible, and experiencing steady flow at a given radial position, we can use Bernoulli's equation to express energy conservation at a given radial position r and height z:

$$\frac{1}{2}mv^2 + mgz = (\text{const}) \tag{42}$$

Since we know $v \propto 1/r$, this means that $z \propto 1/r^2$, and the shape of our draining water surface thus obeys the curve $z(r) = C/r^2$.

4 Viscosity

Adding a viscous (drag) force to our equation of motion is not much of a problem, *in principle.* If we have a viscous force $\mathbf{f}_v$ per unit volume, then Newton's law yields

$$\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} + \rho\frac{\partial\mathbf{v}}{\partial t} = -\nabla P - \rho\nabla\phi + \mathbf{f}_v \tag{43}$$

We simply have to add in the net viscous force per unit volume to the forces due to a pressure gradient and a height variation. The problem is, how do we model the viscous force?

Our model of fluid flow thus far basically ignores the presence of any resistive forces, or forces perpendicular to the direction of the fluid flow. In other words, our first model assumes that the fluid will put up no resistance to being pushed around, which is clearly unrealistic. This is not even realistic for a solid: when we deform a solid, we know that it will produce a restoring force proportional to the strain it experiences, giving rise to Hooke's law macroscopically. Real fluids will also react to an applied force or pressure, but more important in this case than the *amount* of strain is the *rate* at which strain is produced. For example, in most fluids it is easier to move slowly than it is to move rapidly – think about swimming or stirring a jar of thick syrup.

Perhaps more importantly, when we consider a continuous substance like a fluid we have to consider lateral or *shear* forces tangential to the direction of motion. If you stir a jar of syrup along one direction, there is clearly a fluid flow along the directions *tangential* to the motion, meaning there must be forces not just parallel to the fluid motion but in the transverse directions as well. This a type of force you have probably not encountered before, and one which requires careful treatment.

Thinking about the problem another way, our previous model also ignored any interactions between a moving fluid and a solid surface it encounters. In fact, it would have all but impossible to do so without some sort of empirical guidance or at least a hint at the answer. In this, we are lucky, however. One important experimental fact severely constrains models of viscous forces: *the velocity of a fluid is exactly zero at the boundary of a solid surface.* This is not an obvious fact, but one you can easily verify: how else would your ceiling fan blades have dust on the top of them? Shouldn't it blow off?

With this fact, we can attempt a model of viscous forces. Image that we have two flat parallel plates of area A immersed in an initially stationary fluid, separated by a distance d. We hold one plate at rest in the fluid, and move the other plate at velocity v_o through the fluid. In a fluid without viscosity, the moving plate would not disturb the fluid at all, and the fluid velocity would be zero everywhere. However, if the relative velocity of fluid and plate must be zero at each plate's surface, that means that the fluid velocity varies from 0 to v_o moving from the stationary to the moving plate! At the surface of the moving plate, the fluid must have velocity v_o, and at the surface of the stationary plate, it must have $v\!=\!0$.

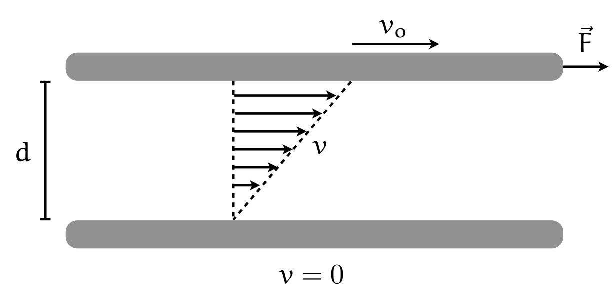


Figure 3: *Viscous drag between two parallel plates in a fluid.*

If you measure the force required to keep the top plate moving, it turns out to be proportional to the velocity of the plate and its area divided by the spacing between the plates at low velocities (low Reynold's numbers).

$$F = \eta \frac{v_o A}{d} \tag{44}$$

The constant of proportionality η is known as the coefficient of viscosity, and to some extent it is a measure of how much force must be supplied to produce motion in a fluid. Noting that power is $\mathbf{F} \cdot \mathbf{v}$, you can see that the power required to maintain a speed v in a fluid scales as ηv^2. What is interesting about this relationship is that the force, along the axis of motion, depends on the *transverse* area of the plate. This is quite different from any force we've encountered so far!

In moving beyond a single dimension, we also have our previous and related problem to consider: if we press on a fluid in one direction, it will move in *all* directions, not just along the direction of the applied force. This is in sharp contrast to our usual considerations of infinitesimal particles, or rigid objects. What do we do when the object can "squish?" In the example above, the moving plate will displace the fluid it is moving through, imparting velocity in the directions perpendicular to the motion of the plate. Evidently, what we lack is a way to relate an applied along one direction with an induced force along another. The mathematical tool we are missing is the *tensor*, a generalization of vectors and scalars which turns out to be indispensable for many areas of physics.

5 Tensors

So far as we need them, a tensor is a set of numbers (or a matrix) that when multiplied by a vector gives back a new vector. Of course, this much can be accomplished by vector or scalar multiplication, but what makes tensors special is that the two vectors need not be simply parallel or perpendicular. This is exactly what we need to understand stress and pressure in materials: relating a displacement or force in one direction to a resulting force along a different direction, particularly when materials are allowed to deform. A close analogy to the type of mathematical object we need is the rotation matrix: a multiplying a given vector by a rotation matrix gives a new vector of the same length, but pointing in a new direction. A tensor is in a sense a more general type of matrix, in which both the length and direction of the resulting vector are generally different.

As an example, let's say we want to consider the conductivity of a material, σ. Ohm's law states that the current density is proportional to the electric field via the conductivity:

$$\mathbf{j} = \sigma \mathbf{E} \tag{45}$$

In an isotropic, homogeneous material, σ is just a scalar, a plain number, that characterizes how much current density results from a specific electric field. As such, a scalar conductivity results in a current density which is always parallel to the electric field. In many materials, this is a perfectly reasonable assumption. However, this is clearly a simplification: what about crystals? In a perfect crystal, we have a symmetric arrangement of atoms which is clearly not isotropic, and it is unphysical to expect the conductivity to be the same along every direction in the crystal. If we have a simple cubic crystal, it would be reasonable to expect the conductivity to be the same along all three crystallographic directions, but we would certainly expect a different conductivity along other directions.

Consider a simple two-dimensional crystal, with a square grid of atoms along the x and y directions. If the spacing of atoms along x and y is the same, we expect that a given electric field applied along the x or y direction would lead to the same current density. However, if we applied the electric field along the line $y = x$, $45°$ with respect to the rows of atoms, we should expect a different current density. Thus, at the very least, our conductivity must be direction-dependent so long as the crystal is not isotropic! Moreover, this means that we can't even reasonably expect that the current density is along the same *direction* as the electric field. If the field is along the the line $y = x$, and we have different conductivities in the x and y directions, we should expect that the resulting current density has both x and y components, and they will not be the same. Even in an isotropic material we have to worry about this to an extent, current will spread out in all directions in a uniform conductor.

In general, the conductivity actually has *nine* components relating electric field to current density, since we have three directions for E combined with three components for j. The conductivity, then, is really a *matrix*:

$$j_i = \sum_j \sigma_{ij} E_j \qquad \text{or} \qquad \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{46}$$

The nine components of σ make a *tensor*, relating $\mathbf{E}$ along an arbitrary direction to a resulting $\mathbf{j}$ along a different direction. The indices of σ signify which component of $\mathbf{E}$ is being related to which component of $\mathbf{j}$: the first index is the component of $\mathbf{E}$, the second the component of $\mathbf{j}$. Incidentally, the fact that we require two indices to tabulate all of the components of σ makes it a "second rank" tensor.[vii] One common notation to indicate that σ is a tensor is σ_{ij}, another is $\overleftrightarrow{\sigma}$, and a third is $\overline{\overline{\sigma}}$, depending on the field of study. We will use σ_{ij}. Thus, for the x component of $\mathbf{j}$, we have

$$j_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z \tag{47}$$

If we apply an electric field purely along the x direction, $\mathbf{j}$ still has components in all three directions due to the anisotropic nature of our crystal:

$$j_x = \sigma_{xx} E_x \qquad j_y = \sigma_{yx} E_x \qquad j_z = \sigma_{zx} E_x \tag{48}$$

Usually, we don't need to deal with all nine components, and we can make use of symmetry to reduce the number of independent components. For instance, the conductivity tensor is symmetric, meaning that $\sigma_{ij} = \sigma_{ji}$. Applying an electric field E_x along the x axis leads to a current density j_y along the y axis, and if we apply the same electric field along the y axis, we end up with the same current density along the x axis:

$$j_y = \sigma_{yx} E_x \qquad j_x = \sigma_{xy} E_y \qquad \sigma_{xy} = \sigma_{yx} \tag{49}$$

In fact, it is possible to simplify the conductivity even further. Our choice of axes along which to decompose the electric field and conductivity vectors, and thus the conductivity tensor, was completely arbitrary. For a second-rank tensor like conductivity, it is always possible to choose axes

[vii] By the same logic, we can call vectors "first rank" tensors, needing only one index, and scalars "zero rank" tensors.

such that the tensor is *diagonal*, e.g., such that only σ_{xx}, σ_{yy}, and σ_{zz} are non-zero:

$$\begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{50}$$

In a crystal, finding the diagonal representation of a tensor typically corresponds to choosing the natural crystallographic axes for decomposing the electric field and current density. Finally, in an isotropic material, in which the conductivity is independent of direction, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$, and we may treat the conductivity as a simple scalar.

5.1 Conductivity tensor for the Hall effect

Actually, we already know one simple situation in which a tensor conductivity is required *even for an isotropic medium*: the Hall effect.[viii] You used tensors without even knowing it! Imagine we have a sheet of uniform conductor lying in the xy plane, with an electric field E_y applied along the y axis. This will impart a velocity, on average, of v_y to each positive charge q in the conductor. In the absence of a magnetic field, Ohm's law plus a free-electron model gives us the current density in the y direction:

$$j_y = \sigma_{yy} E_y \qquad \text{with} \qquad \sigma_{yy} = \frac{nq^2\tau}{m} = \mu n q \tag{51}$$

Here n is the number of charge carriers per unit volume, m the mass and q the charge per carrier, τ the average time between collisions, and $\mu = q\tau/m$ the carrier mobility. Now, of course, we know that we need to label σ as a tensor. In this case, we need the component σ_{yy} when both current density and electric field are along y. Applying an electric field along the x direction for our homogenous sample would lead us to $\sigma_{xx} = \sigma_{yy}$. If there is no magnetic field, then this is the end of the story: $\sigma_{xy} = \sigma_{yx} = 0$, since there is no net force on the charges in the directions perpendicular to E. Thus, our conductivity tensor is diagonal, and the diagonal elements are all the same, which means we can treat the conductivity as a simple scalar. No problem.

Next, we add a magnetic field B_z in the $+z$ direction in addition to the electric field in the y direction. Now we have a transverse magnetic force on the charge carriers, $F_B = qv_yB_z$, acting on positive charges in the $+x$ direction. This leads to a separation of charge along the x axis, which means there must be an electric field $-E_x$ now. This electric field will give rise to a force opposing the charge separation: the stronger the magnetic field, the stronger the magnetic force, the larger the charge separation, but the larger the electric field. At equilibrium, the two horizontal forces must be equal, $qE_x = qv_yB_z$, or $E_x = v_yB_z$. We could have arrived at this result far more quickly using the relativistic transformations of the fields: in the charges' reference frame, traveling at velocity $\mathbf{v}$,

[viii] If you haven't had electricity and magnetism, you can skip this section.

the magnetic field appears as an electric field of magnitude $\mathbf{E}' = \mathbf{v} \times \mathbf{B}$.

This is the usual Hall effect you have seen in electromagnetism: a magnetic field applied orthogonal to a current gives rise to an electric field (or potential difference) along the third axis. What it means is that now we have a "shear" component to our conductivity tensor: the electric field and current density along y in the presence of a magnetic field along z gives us an electric field along x! Even though the conductor itself is isotropic, the presence of a magnetic field breaks the symmetry of the problem, and that is sufficient to require a tensor to relate $\mathbf{j}$ and $\mathbf{E}$.

Now we just need to figure out what the new component of the conductivity is. Since in the field B_z we have E_x resulting from j_y, we can guess that it must be σ_{yx}. We have already related E_x and B_z, we can relate E_x and j_y by noting the relationship between current density and drift velocity: $j_y = nqv_y$. Thus,

$$E_x = v_y B_z = \frac{j_y B_z}{nq} \tag{52}$$

$$j_y = \frac{nq}{B_z} E_x = \sigma_{yx} E_x \qquad \Longrightarrow \qquad \sigma_{yx} = \frac{nq}{B_z} \tag{53}$$

What about σ_{yx}? If we were to apply the electric field along the x direction instead, but keep B in the z direction, we would have a current density along the x direction, but the induced electric field would be along $-y$ rather than $+y$. This means $\sigma_{yx} = -\sigma_{xy}$. In the presence of orthogonal electric and magnetic fields, we can now write down the entire conductivity tensor and the relationship between $\mathbf{j}$ and $\mathbf{E}$

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \mu nq & \dfrac{nq}{B_z} \\ \dfrac{-nq}{B_z} & \mu nq \end{bmatrix} \qquad j_i = \sum_{j=1}^{3} \sigma_{ij} E_j \tag{54}$$

This conductivity tensor encompasses both normal Ohmic conduction and the Hall effect. If the fields are not orthogonal, this is not much of a problem, at least in two dimensions, since only the component of $\mathbf{B}$ orthogonal to $\mathbf{E}$ will alter the conductivity in this simple picture.

5.2 Other examples of tensors

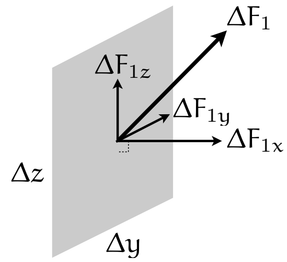
In fact, you've already encountered tensors many times, probably without knowing it. Generally speaking, if you need to relate two vectors, and they in general need not be strictly parallel or perpendicular, a tensor is probably involved. For instances, the moment of inertia is really a 2nd-rank tensor, since angular momentum and angular velocity are not in general parallel. Torque is also a 2nd-rank tensor, and *anti-symmetric* ($\tau_{ij} = -\tau_{ji}$), but happens to transform like a vector in three dimensions. For that reason, we usually just treat it as a vector (or pseudovector, really) since we can get away with it!

6 The Stress Tensor

So what is stress? Essentially, it is nothing more than a generalization of pressure, a net force per unit area. Hydrostatc pressure we are used to dealing with is just a special type of stress, when the net force is normal to area of consideration. In a static fluid, the force on each side of an infinitesimal cube of fluid is the same in magnitude and always normal to the surfaces of the cube. In this case, the stress is just the hydrostatic pressure, and it is a simple scalar: $\mathbf{F} = PA\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector normal to the area A.

When we wish to deal with the internal forces in continuous objects, however, this need not be true. Inside a solid object or a fluid, we know there are internal forces between neighboring parts of the material holding it together. Consider first a cube of a nice squishy substance like gelatin, and cut it into two pieces. Clearly, before we cut the gelatin, there must have been a force holding the two pieces together. Before the cut, each half exerted a force ΔF on the other to hold the block of gelatin together, so the stress in the material was simply this force divided by the area of the cut surface. However, the net force between the two pieces was not simply perpendicular to the cut surface. If that were true, any infinitesimal force along the cut plane would have separated the two pieces. Thus, there must be forces acting not just normal to any surface in the block, but also along the two tangential directions. In order to properly treat a patch of surface within a continuous object, we must deal with all three components of force acting on the surface. This is what stress is, a generalization of pressure to encompass forces acting on a surface in all three directions.

Let us go back to a simple example, where we have a flat plate moving at velocity v_o through a fluid. In that case, we had two types of forces present. First, we had a force per unit area on the surfaces of the plate due to the hydrostatic pressure of the fluid, which acted equally in all directions and normal to each surface. This force can be described by a simple scalar, the pressure, and the area of the plate. Second, we had a force acting antiparallel to the velocity due to the viscous drag of the fluid. This is what we would call a shear force, being tangential to the surface of the plate. The force per unit area due to viscous drag is thus a *shear stress*, acting in the $-x$ direction, and it depends on a velocity in the x direction and an area in the xy plane. A complete description of such forces will require a tensor, the *stress tensor*. As another quick example, let's go back to our cube of gelatin. Say we press down (along $-z$) on the upper face lying in the xy plane. This will clearly lead to a net force in the z direction on both faces in the xy plane, and a net shortening of the cube along the z direction. If the gelatin is incompressible, however, conservation of matter requires that the cube bulge out in the x and y direction, meaning there must be outward forces on the other four faces of the cube! Again, we will need a tensor to describe this situation, since we have an applied force in one direction leading to net forces in all three directions.



$$\Delta F_{1y} = \tau_{yx}\Delta y \Delta z \qquad \tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Figure 4: *Force components on a planar slice of fluid with its area normal along* $\hat{\mathbf{x}}$.

How can we figure out what the stress tensor looks like? Let's consider a volume of continuous incompressible material, of constant density ρ. Now, take a small slice of this material perpendicular to the x axis, making a little square of sides Δy and Δz with area normal $\hat{\mathbf{x}}$, shown in Fig. 4. If we apply a force $\mathbf{\Delta F}_1$ to this surface along an arbitrary direction, we can break it up into components ΔF_{1x} normal to the surface and ΔF_{1y} and ΔF_{1z} tangential to the surface. The components of stress are just these forces divided by the area of our surface, labeled with two indices: the first labeling the direction of the force component, the second the area normal. For example, the force per unit area along the y direction is just

$$\tau_{yx} = \frac{\Delta F_{1y}}{\Delta y \Delta z} = \frac{\Delta F_{1y}}{\Delta a_x} \tag{55}$$

where Δa_x is just the area of our element of surface perpendicular to the x direction. Similarly, we have net forces per unit area in the x and z directions,

$$\tau_{zx} = \frac{\Delta F_{1z}}{\Delta y \Delta z} \qquad \tau_{xx} = \frac{\Delta F_{1x}}{\Delta y \Delta z} \tag{56}$$

The stress τ_{xx} acts normal to our little area, just as a simply hydrostatic pressure would, while the stresses τ_{yx} and τ_{zx} act along the transverse directions. In total, just to describe the stress along a single axis, we need *three* components, which means that a full description of all the stresses on an object will require *nine* components. For example, if we now take a slice of material lying perpendicular to the y axis, lying in the xz plane, this area will have a net force $\mathbf{\Delta F}_2$ acting on it, and resolving it along the three axes leads to stresses τ_{xy}, τ_{yy}, and τ_{zy}. We can make a similar construction for a slice perpendicular to the z axis, and in total the stress on our object will be characterized by nine numbers, which we can conveniently express as a matrix:

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{57}$$

The diagonal components τ_{ii} are normal stresses, representing forces per unit area acting perpendicular to the area of a given face. These components are what we would usually just call pressure, the net force per unit area acting perpendicular to a given face. The off-diagonal components are the *shear stresses* acting along the two directions tangential to a given face, analogous to the tangential frictional force present when we slide two objects past one another. Our nine numbers τ_{ij} in total make up the *stress tensor*, where i indicates the direction along which the stress acts, and j indicates the surface normal of the relevant face. Thus, τ_{xy} represents a shear stress acting in the x direction on a face whose area normal points in the y direction.

At this point, it is probably useful to draw a little picture. Take a small cube of material, aligned along the x, y, and z axes. Looking down the z axis at one side of the cube, we can visualize the components of the stress in the x and y directions acting on four of the faces, shown in Fig. 5.

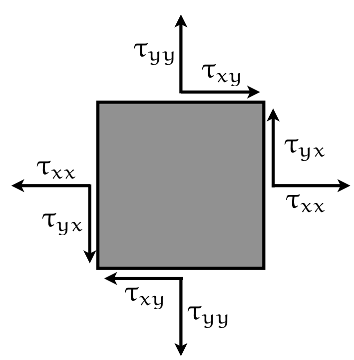


Figure 5: *The forces in the x and y directions on the faces of an infinitesimal cube. The diagonal components of stress τ_{ii} act normal to each face, while the off-diagonal components τ_{ij} act tangentially to each face. Since the cube is very small, the stresses do not change appreciably across the cube.*

As you can see, the forces on real continuous objects are rather complicated. From our initial discussion of a static fluid, requiring only a simple scalar pressure, we now have a nine-component second-rank tensor. However, it is not as bad as it seems: the stress tensor turns out to be symmetric, and we don't need all nine components. If we consider an infinitesimally small cube of material, then we can imagine that the stresses do not change appreciable from one side to the other. As shown in the figure above, the forces on opposing sides of the cube must be equal and opposite in this case. This also implies that the *torque* about the center of the cube is zero – if it were not, the cube would start spinning, which would be unphysical for an infinitesimally small object. If our cube has sides of dimension Δa, then we can easily write down the torque about the center as $\Delta a(\tau_{yx} - \tau_{xy}) = 0$, which means $\tau_{xy} = \tau_{yx}$. We can apply the same logic looking at the other faces of the cube, and just by considering that the cube must be in rotational equilibrium we find that the stress tensor must be symmetric, $\tau_{ij} = \tau_{ji}$. In short, we have only have six unique components of stress, rather than nine.

Since our stress tensor is symmetric, it can be described by a symmetric matrix. If you have taken linear algebra, you might recall that this leads us to an even more important property of the stress tensor: since it is symmetric, it is always possible to find a choice of coordinate axes for which it is diagonal. That is, if we choose our coordinate axes carefully, it is always possible to find a special orientation for which our stress tensor has only the three components τ_{xx}, τ_{yy}, and τ_{zz}. In a perfect crystalline material, this special choice may correspond to the crystallographic axes, for instance. However, in general, the stress tensor varies from point to point in a material, meaning it is actually a tensor *field*. Just like we have a scalar field $T(x, y, z)$ describing the temperature everywhere in a room, or a vector field $\mathbf{E}(x, y, z)$ describing the electric field through all space, our stress tensor field describes the components of stress at all points in a material. At every point in space, the stress tensor gives us nine numbers – six unique numbers – describing the forces at that point, and thus a full description of the forces in a body require six functions of position.

6.1 The Maxwell stress tensor

Incidentally, there is a stress tensor associated with *electromagetic* forces per unit volume, the *Maxwell stress tensor.* The electromagnetic force per unit volume can be written in terms of the electric and magnetic fields along with the charge and current densities

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{58}$$

Though it is quite some work to prove it, one can also write the electromagnetic force per unit volume as the divergence of a second-rank tensor:

$$\mathbf{f} = \nabla \cdot T \tag{59}$$

Here T is the Maxwell stress tensor, and it is defined by

$$T_{ij} = \epsilon_o \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_o} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{60}$$

Where δ_{ij} is the Kronecker delta function, something which shows up quite a bit in tensor and vector analysis. It has a very simple definition:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{61}$$

Written out explicitly as a matrix,

$$T = \frac{1}{4\pi} \begin{bmatrix} E_x^2 - \frac{E^2}{2} & E_x E_y & E_x E_z \\ E_x E_y & E_y^2 - \frac{E^2}{2} & E_y E_z \\ E_x E_z & E_y E_z & E_z^2 - \frac{E^2}{2} \end{bmatrix} \tag{62}$$

As with our general stress tensor in the previous section, the diagonal T_{ii} components give rise to pressure-like forces, while the off-diagonal T_{ij} give rise to shear forces. You are certain to encounter this tensor again in later physics courses ...

7 Viscosity and stress in three dimensions

After a long detour, we can finally return to our parallel plates moving within a fluid from Sect. 4. Recall our setup:

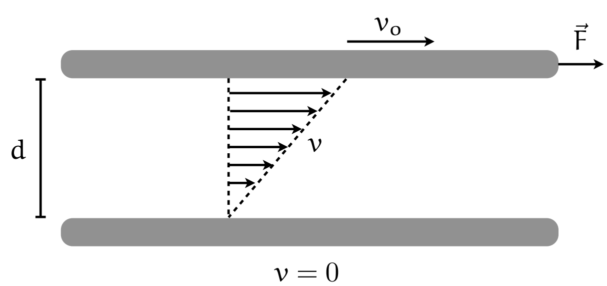


Figure 6: *Viscous drag between two parallel plates in a fluid.*

To be a bit more concrete, let the x axis be in the direction of the applied force, the y direction upward, and the z direction out of the page. Using our new tensor machinery, we can write the force per unit area along the x direction required to keep the top plate moving as a stress:

$$\tau_{yx} = \frac{F_x}{\Delta y \Delta z} = \eta \frac{v_o}{d} \tag{63}$$

Here we used Eq. 44, and noted that $A = \Delta y \Delta z$. Thus, what we have previously considered is a shear stress acting tangential to the plates due to a viscous force along the x axis, and empirically it is found to be proportional to the speed of the upper plate through a scalar coefficient we call the viscosity.

As a slightly more general case, we could forget about the plates, and only consider artificial surfaces within the fluid itself, moving at different velocities. In Fig. 7 below, we look at an extended region within a moving fluid, with its faces parallel to the fluid flow. In general, the velocity of the fluid will vary along the vertical direction (y), such that at the top of our cell the velocity is $v+\Delta v$, and at the bottom it is just v. Based on our simpler model above, the net shear force acting on the cell in the x direction will the difference between the forces on the top and bottom of the cell, divided by the area of the plate ΔA. By analogy with the situation with two plates, the shear stress is then proportional to the difference in velocity between the top and bottom of the cell divided by the vertical extent of the cell:

$$\frac{\Delta F_x}{\Delta A} = \eta\frac{\Delta v}{\Delta y} = \eta\frac{\partial v_x}{\partial y} = \tau_{yx} \qquad \text{(fluid velocity along } x \text{ only)} \tag{64}$$

This net shear stress acts to the right on the top face, and would tend to either deform our cell into a parallelogram or lead to a rotation of our cell in the clockwise direction. The only way that the fluid will be irrotational is if $\partial v_x/\partial y = 0$, that is, the velocity is constant along the vertical direction and there is no net force at all. Otherwise, a variation in fluid velocity along the *vertical* direction leads to a *horizontal* shear stress.

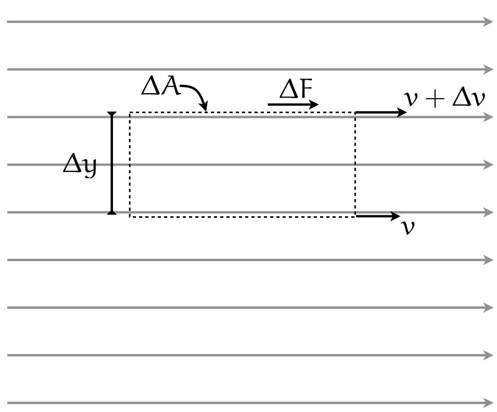


Figure 7: *A small volume of fluid within a flow.*

What if the velocity of the fluid isn't strictly parallel to the faces of our cell? Let's say the vertical component of the velocity varies across the top and bottom faces, with the velocity being higher on the left side of the cell. This situation would also tend to cause a clockwise rotation of our cell, meaning that there must also be stress components along the x direction due to the variation of fluid velocity in the x direction (as well as normal components along the y direction). For a general fluid velocity, the horizontal shear stress must then have two components:

$$\tau_{yx} = \eta \frac{\partial v_y}{\partial x} + \eta \frac{\partial v_x}{\partial y} \tag{65}$$

We could find the other shear components τ_{yz} and τ_{zx} similarly. Note that this equation immediately satisfies our symmetry requirement $\tau_{xy} = \tau_{yx}$. We can also see from this general expression that there are only three cases in which there is *no* shear stress in the fluid: either the fluid is static ($\mathbf{v}=0$), the fluid's velocity varies only along the out-of-plane z direction ($\partial v_x/\partial y = \partial v_y/\partial x = 0$), or the fluid is uniformly rotating ($\partial v_x/\partial y = -\partial v_y/\partial x$). Of course, there are also no shear forces in a fluid with zero viscosity, but such things are incredibly rare.[ix]

Along these same lines, we can also find the normal stresses, those acting perpendicularly to the faces of our cell. For example, if there is a variation in velocity along the vertical direction, then there will also be a net force on the cell in the vertical direction, along with the shear component:

$$\tau_{yy} = 2\eta \frac{\partial v_x}{\partial x} = \frac{F_y}{A} = P \tag{66}$$

Here A is the area of the cell, and this tells us that if $\partial v_x/\partial x$ is nonzero (e.g., the velocity varies along the flow direction), there is a pressure change along that direction as well. The normal components of the stress, τ_{ii}, are what we would simply call pressure if the fluid were static. In the case of a moving fluid, the total normal force per unit area would be the static pressure P plus the normal stress due to the fluid motion.

In general, *so long as the fluid is incompressible*, based on our description above you should be able to convince yourself that the stress components are given by

$$\tau_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{67}$$

8 Viscous forces in three dimensions

Now that we have the shear stresses in the fluid in the presence of viscosity, we can complete our equation of motion. All we need to do is work backwards to determine the forces on an arbitrary cell within a moving fluid from the stress components. Imagine we have again a small cube of fluid, Fig. 8, whose faces are aligned with our coordinate axes, with sides of length Δx, Δy, and Δz. In addition to any hydrostatic pressure, our cube will have a force on each of its six sides due to the stress in the moving fluid fluid, whose velocity we will assume to vary in magnitude and direction.

[ix] Liquid helium is a so-called "superfluid" with zero viscosity, a macroscopic quantum-mechanical effect that can be observed only at very low temperatures.

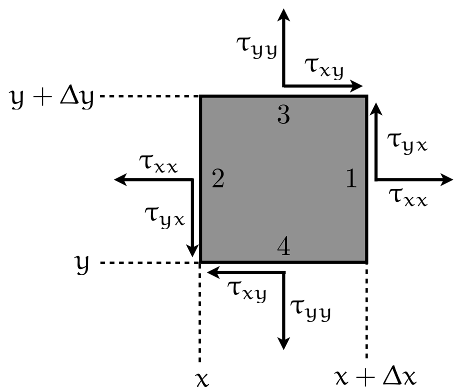


Figure 8: *The stresses in the x and y directions on the faces of a cube cube of fluid.*

First, we can tabulate all of the forces along a given axis, starting with x. All six faces of the cube will have a stress component in the x direction: four shear forces, and two normal forces. On face 1, we have a normal stress component τ_{xx} acting over an area $\Delta y \Delta z$, and net x component of the force on face 1 will be the product of stress and area. However, we must be careful: the stress is really a tensor *field*, and it varies with position, so the value of τ_{xx} is different for face 1 and face 2, for instance. Thus, we should explicitly note at which position we are evaluating the stress components. With that in mind,

$$F_{x1} = \tau_{xx}\Big|_{x+\Delta x} \Delta y \Delta z \tag{68}$$

The x component force on face two will be similar and opposite in sign, the only substantial difference is that we are evaluating the stress tensor at a different position:

$$F_{x2} = -\tau_{xx}\Big|_{\Delta x} \Delta y \Delta z \tag{69}$$

Faces 3 and 4 will also have forces in the x direction through the shear stress τ_{xy}. Face 3 has area $\Delta x \Delta z$ and it is located at vertical position $y + \Delta y$. Face 4 has the same area, but the force is in the opposite direction and τ_{xy} should be evaluated at a vertical position y.

$$F_{x3} = \tau_{xy}\Big|_{y+\Delta y} \Delta x \Delta z \tag{70}$$

$$F_{x4} = -\tau_{xy}\Big|_{y} \Delta x \Delta z \tag{71}$$

$$\tag{72}$$

Finally, faces 5 and 6 (on the front and back of the cube in Fig. 8) also have forces along the x direction through the shear stress τ_{xy}:

$$F_{x5} = \tau_{xz}\Big|_{z+\Delta z} \Delta x \Delta y \tag{73}$$

$$F_{x6} = -\tau_{xz}\Big|_{z} \Delta x \Delta y \tag{74}$$

$$\tag{75}$$

All that is required now is to tabulate the net force along the x direction for the whole cube:

$$F_x = F_{x1} + F_{x2} + F_{x3} + F_{x4} + F_{x5} + F_{x6} \tag{76}$$

$$= \left(\tau_{xx}\Big|_{x+\Delta x} - \tau_{xx}\Big|_{x}\right)\Delta y \Delta z + \left(\tau_{xy}\Big|_{y+\Delta y} - \tau_{xy}\Big|_{y}\right)\Delta x \Delta z + \left(\tau_{xz}\Big|_{z+\Delta z} - \tau_{xz}\Big|_{z}\right)\Delta x \Delta y \tag{77}$$

$$= \Delta\tau_{xx}\Delta y \Delta z + \Delta\tau_{xy}\Delta x \Delta z + \Delta\tau_{xz}\Delta x \Delta y \tag{78}$$

As with our equation of motion without viscosity, it is more useful to consider the force per unit volume along x, which we'll call f_x:

$$f_x = \frac{F_x}{\Delta x \Delta y \Delta z} = \frac{\Delta\tau_{xx}}{\Delta x} + \frac{\Delta\tau_{xy}}{\Delta y} + \frac{\Delta\tau_{xz}}{\Delta z} \tag{79}$$

If we take the limit that the dimensions of our cube become infinitesimally small, what we have is the definition of a partial derivative:

$$f_x = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} \tag{80}$$

We can repeat the analysis for the forces along the other directions, and our general expression for an incompressible fluid is

$$f_i = \sum_{j=1}^{3} \frac{\partial\tau_{ij}}{\partial x_j} \qquad \text{with} \qquad \tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{81}$$

The stresses in a fluid depend on the velocity gradients in the fluid (or, equivalently, the rate of change of shear strain), while the *forces* depend on the *stress gradients*. There will only be a net force on a volume of fluid if there is a net spatial variation of stress, and there will only be stress if there is a net spatial variation in velocity. Combining the two relationships above, we can cut out the middleman and relate the viscous force per unit volume directly to the velocity distribution:

$$f_i = \eta \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \tag{82}$$

If we write out all three components of the force and rearrange the terms, we can recover a much more compact vector equation. Let's rearrange the sum and see what comes out:

$$f_i = \eta \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] = \eta \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} + \eta \frac{\partial}{\partial x_i} \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = \eta \nabla^2 v_i + \eta \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v} \tag{83}$$

Considering all three components of the force per unit volume, we have a nice vector equation in the end:

$$\mathbf{f} = \eta \nabla^2 \mathbf{v} + \eta \nabla \left(\nabla \cdot \mathbf{v} \right) \tag{84}$$

Here we have used the vector Laplacian $\boldsymbol{\nabla}^2 \mathbf{v}$, which is just $\nabla^2 v_x \hat{\mathbf{x}} + \nabla^2 v_y \hat{\mathbf{y}} + \nabla^2 v_z \hat{\mathbf{z}}$. We can make this still simpler, however, by remembering that for an incompressible fluid (which we have already assumed!), the continuity equation reads $\nabla \cdot \mathbf{v} = 0$. With that in mind, after much pain we ultimately have a simple form for the viscous force in an incompressible fluid

$$\mathbf{f}_v = \eta \boldsymbol{\nabla}^2 \mathbf{v} \qquad \text{(viscous force, incompressible fluid)} \tag{85}$$

Our pervasive assumption of an incompressible fluid does come at a price. For instance, we will not be able to treat density variations in the fluid, such as sound waves. However, a wide variety of interesting fluids are essentially incompressible, and the form of the viscous force per unit volume above is sufficient. This assumption serves quite well for, e.g., air flowing at low speeds compared to the speed of sound.

9 The equation of motion with viscosity for incompressible fluids

The general equation of motion we developed was

$$\rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla P - \rho \nabla \phi + \mathbf{f}_v \tag{86}$$

where f_v was our yet-to-be-determined viscous force. For an incompressible fluid, using the form of the viscous force from Eq. 85, our equation of motion reads

$$\rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla P - \rho \nabla \phi + \eta \boldsymbol{\nabla}^2 \mathbf{v} \tag{87}$$

This non-linear partial differential equation is the Navier-Stokes equation for flow of Newtonian incompressible fluids. The Navier-Stokes equation is not straightforward to interpret qualitatively, and famously difficult to solve in even the simplest cases. Another more common form substitutes the potential per unit mass $\nabla\phi = -\mathbf{g}h$, where h is the change in height:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla P + \rho\mathbf{g}h + \eta\boldsymbol{\nabla}^2\mathbf{v} \tag{88}$$

This form is more easily interpreted as a statement of Newton's law: mass (ρ) times acceleration equals the sum of forces, namely pressure ($-\nabla P$), viscous forces ($\eta\nabla^2\mathbf{v}$), and gravity ($\rho\mathbf{g}$). Since we are assuming constant density (incompressible fluid), the continuity equation is simply $\nabla\cdot\mathbf{v} = 0$, which is a statement of conservation of fluid volume.

Incidentally, we can also reintroduce our vorticity $\boldsymbol{\Omega} = \nabla\times\mathbf{v}$:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\Omega}\times\mathbf{v} + \frac{1}{2}\nabla v^2\right) = -\nabla P + \rho\mathbf{g}h + \eta\boldsymbol{\nabla}^2\mathbf{v} \tag{89}$$

That means that for an irrotational fluid ($\boldsymbol{\Omega} = 0$), we have

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2\right) = -\nabla P + \rho\mathbf{g}h + \eta\boldsymbol{\nabla}^2\mathbf{v} \tag{90}$$

The steady-state ($\partial\mathbf{v}/\partial t = 0$) equation for an irrotational fluid reads

$$\frac{1}{2}\rho\nabla v^2 = -\nabla P + \rho\mathbf{g}h + \eta\boldsymbol{\nabla}^2\mathbf{v} \tag{91}$$

9.1 Steady flow through a long cylindrical pipe

Certainly after all this mess we should be able to handle the steady flow of water through a pipe. Let's try it out. We will take a very long circular pipe of length L and radius R whose axis is oriented along the z direction, and it is carrying an incompressible fluid of density ρ. Clearly, it will be convenient to work in cylindrical coordinates. We will imagine that we set up a pressure difference between the end points to establish a flow of fluid, such that we have a pressure P_L at $z = L$ and P_o at $z = 0$. If the pipe is very long and we can ignore the entrance and exit effects, then the pressure gradient along the length of the pipe is just

$$\frac{\partial P}{\partial z} = \frac{P_L - P_o}{L} \tag{92}$$

Along the radial direction, there must be no pressure gradient if we have uniform flow of an incompressible fluid, $\partial P/\partial r = 0$. We can also establish some symmetry and boundary conditions on the velocity of the fluid in the pipe. Due to the symmetry of the problem, the velocity is independent of θ. Further, if the pipe is very long, then the radial component of the velocity v_r should be independent of z (in fact, it should be zero!). We need only worry about the variation of v_z with r, since v_z should also be independent of z for a very long pipe – for an incompressible fluid, continuity ensures that the velocity of the fluid cannot vary along the length of the pipe. Finally, at the pipe boundary, we can also enforce the prior condition that the fluid is static, such that $v(R)=0$. This condition means that the v_z must peak at the center of the pipe, since it is zero at both edges, and thus $\partial v_z/\partial r$ must vanish at the center of the pipe, $r=0$. The problem that remains is to deduce what the variation of velocity along the axis of the pipe.

If we are only interested in slow and steady flow through the pipe (small Reynolds' numbers) of an incompressible fluid, we may also neglect the "acceleration" term in the Navier-Stokes equation $\frac{1}{2}\rho\nabla v^2$. Presuming the ends of the pipe differ in height by h,

$$-\nabla P + \eta\nabla^2\mathbf{v} + \rho\mathbf{g}h = 0 \tag{93}$$

With this convention, for positive h the end of the pipe at $z=L$ is h above the end at $z=0$. This is just one step up from our equation of state for a completely static fluid, we now retain only the viscous force $\eta\nabla^2\mathbf{v}$. Neglecting the acceleration is just saying we wish to find a solution for which the velocity is constant, the definition of steady flow. Since we may also neglect the variation of velocities along the $\hat{\boldsymbol{\theta}}$ direction, in cylindrical coordinates we have a simplified Navier-Stokes equation:

$$-\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r h = 0 \tag{94}$$

$$-\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z h = 0 \tag{95}$$

If we assume that the gravitational force acts along the $-z$ direction only (so that the vertical change in height of the pipe is h along z from one end of the pipe to the other) and apply our boundary/symmetry conditions, we have

$$\frac{\partial P}{\partial r} = \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)\right] = 0 \tag{96}$$

$$\frac{\partial P}{\partial z} = \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] + \rho g_z h = \frac{P_L - P_o}{L} \tag{97}$$

If we rearrange the second equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\eta}\left(\frac{P_L - P_o}{L} - \rho g_z h\right) \tag{98}$$

Multiplying both sides by r and integrating with respect to r, we have

$$r\frac{\partial v_z}{\partial r} = \frac{r^2}{2\eta}\left(\frac{P_L - P_o}{L} - \rho g_z h\right) + C_o \tag{99}$$

where C_o is a constant of integration, to be determined by the boundary conditions. We can pull the same trick twice, this time dividing by r and integrating with respect to r, and arrive at

$$v_z = \frac{r^2}{4\eta}\left(\frac{P_L - P_o}{L} - \rho g_z h\right) + C_o r + C_1 \tag{100}$$

where C_1 is a second constant of integration. We can eliminate C_o by enforcing the condition that at the center of the pipe, $r=0$, $\partial v_z/\partial r$ must vanish. We can find C_1 by enforcing the condition that $v_z(R)=0$:

$$v_z(R) = 0 = \frac{R^2}{4\eta}\left(\frac{P_L - P_o}{L} - \rho g_z\right) + C_1 \qquad \Longrightarrow \qquad C_1 = \frac{R^2}{4\eta}\left(\frac{P_o - P_L}{L} - \rho g_z\right) \tag{101}$$

This fully determines the radial variation of the vertical velocity of the fluid in the pipe:

$$v_z(r) = \left(\frac{P_o - P_L}{L} - \rho g_z h\right)\frac{R^2 - r^2}{4\eta} \tag{102}$$

There are two terms: one due to the applied pressure gradient, the other due to hydrostatic gravity-induced flow. Given this distribution, we could find the volumetric flow rate Q (e.g., cubic meters per second) by integrating $v_z(r)$ over annular cross sections of the pipe. Each such annulus has width dr, and sits at a radius r from the pipe center with r running from 0 to R. The area of each annulus is $dA=2\pi r\,dr$, and over a pipe length $dl=v_z(r)\,dt$ in a time dt, represents a volume $dV=dA\,dl$. The flow rate through this annulus is $dQ=dV/dt=2\pi r v_z(r)\,dr$. Put another way, we find Q by integrating the velocity distribution over the area of annular slices of our pipe of area $2\pi r\,dr$.[x] The volume flow rate is then

$$Q = \int_0^R 2\pi r v_z(r)\,dr = \int_0^R 2\pi r\left(\frac{P_o - P_L}{L} - \rho g_z h\right)\frac{R^2 - r^2}{4\eta}\,dr = \frac{\pi R^4}{8\eta}\left(\frac{P_o - P_L}{L} - \rho g_z h\right) \tag{103}$$

Again, the first term is flow due to the pressure difference, while the second is the flow due to the change in height of the pipe over its length. If h is positive, the far end is higher than the near end, and the gravity flow is from L to 0. If h is negative, the flow is "downhill" from 0 to L. Correspondingly, if $P_o > P_L$ the pressure gradient causes flow from 0 to L, and for $P_L > P_o$, the flow is from L to 0. For a horizontal pipe, or at least one where $\rho g_z h$ is small compared to the pressure gradient, we find

$$Q = \frac{\pi R^4\left(P_o - P_L\right)}{8\eta L} \tag{104}$$

[x] We could also find the average value of $v_z(r)$ over the cross section and multiply it by the area and arrive at the same result.

This is the famous Hagen-Poiseuille law, and the basis for most plumbing you will encounter. The R^4 dependence is dramatic: halving the diameter of the pipe decreases the flow rate by a factor of 16! One can also rearrange this in a more practical way: given a desired flow rate Q, a given fluid of viscosity η, and a pipe of diameter R, what pressure difference $\Delta P = P_o - P_L$ is required? Assuming the pipe is over level ground, we can neglect the $\rho g_z h$ contribution to the flow, and

$$\Delta P = \frac{8\eta l Q}{\pi R^4} \tag{105}$$

This also tells you that the pressure drops uniformly with the length of the pipe, and it scales linearly with viscosity and flow rate.

9.2 Stoke's flow around a solid sphere

Now let us consider the steady flow of an incompressible fluid around a dense, solid sphere, or equivalently, a solid sphere falling in a static fluid. For small spheres, this leads to measurement known as the *falling sphere viscometer*, as we will find that the terminal velocity of the falling sphere will be inversely proportional to the viscosity. As in the example above, in this case we may neglect the "acceleration" term in the Navier-Stokes equation $\frac{1}{2}\rho\nabla v^2$ if we are looking for a steady solution:

$$-\nabla P + \eta\nabla^2\mathbf{v} + \rho\mathbf{g} = 0 \tag{106}$$

Further simplification is possible in this case due to the symmetry of the problem. Let the fluid flow be along the z axis with constant velocity V_∞, with the solid sphere of radius R at the origin. In this case, by symmetry the fluid momentum is clearly independent of φ (the angle in the xy plane). In spherical coordinates the Navier-Stokes and continuity equations read

$$-\frac{\partial P}{\partial r} + \eta\left[\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}v_\theta\cot\theta\right] + \rho g_r = 0 \tag{107}$$

$$-\frac{1}{r}\frac{\partial P}{\partial\theta} + \eta\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\right] + \rho g_\theta = 0 \tag{108}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) = 0 \tag{109}$$

Here we have expanded the θ and r portions of the gradient operators in spherical coordinates. Perhaps surprisingly, the stress distribution, pressure distribution, and velocity components can be found analytically:

$$\tau_{r\theta} = \frac{3}{2}\frac{\eta V_\infty}{R}\left(\frac{R}{r}\right)^4 \sin\theta \tag{110}$$

$$P = P_o - \rho g z - \frac{3}{2}\frac{\eta V_\infty}{R}\left(\frac{R}{r}\right)^2 \cos\theta \tag{111}$$

$$v_r = V_\infty\left(1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta \tag{112}$$

$$v_\theta = -V_\infty\left(1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin\theta \tag{113}$$

Note the following conditions must be met at the sphere's boundary: $r=R$, $v_r=v_\theta=0$, and additionally at $r=\infty$, $v_z=V_\infty$. Equation 111 is readily parseable: P_o is the pressure in the plane $z=0$ far from the sphere, $-\rho g z$ is the hydrostatic pressure effect, and the term with V_∞ results from fluid flow around the sphere. These equations are valid for Reynolds numbers less than approximately one. In Fig. 9, we show fluid flow around a sphere under these conditions.

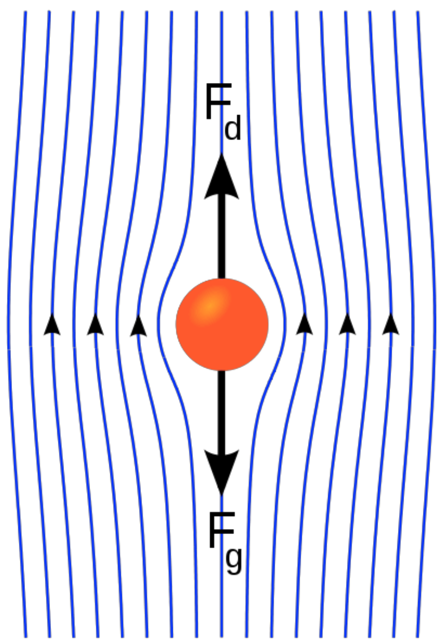


Figure 9: *Forces on and streamlines around a sphere in Stokes flow. From* http://en.wikipedia.org/wiki/File:Stokes_sphere.svg.

What we are interested in now is the force on the sphere due to this flow. The normal force (along the z axis) acting on the solid sphere is due to the pressure given by Eq. 111 with $r = R$ and $z=R\cos\theta$:

$$P(r=R) = P_o - \rho g R\cos\theta - \frac{3}{2}\frac{\eta V_\infty}{R}\cos\theta \tag{114}$$

The net upward force in the z direction due to the pressure difference on the 'top' and 'bottom' portions of the sphere is found by multiplying this pressure times the infinitesimal bit of surface area over which it acts, $R^2 \sin\theta d\theta\, d\varphi$ and integrating over the surface of the sphere:

$$F_n = \int_0^{2\pi}\int_0^{\pi} \left[P_o - \rho g R \cos\theta - \frac{3}{2}\frac{\eta V_\infty}{R}\cos\theta \right] R^2 \sin\theta d\theta\, d\varphi \tag{115}$$

$$F_n = \frac{4}{3}\pi R^3 \rho g + 2\pi\eta R V_\infty \tag{116}$$

We recover two terms for the normal force: the first is the *buoyant force* and the second the *form drag*. At each point on the surface, there is also a shear stress acting tangentially, $-\tau_{r\theta}$. This tangential force, since we are dealing with a curved surface, has both $x-y$ and z components. Over the whole sphere, the former will vanish by symmetry, but the latter will give rise to a net force for any non-zero fluid velocity. On any infinitesimal patch of surface, the z-component of this tangential force is $(-\tau_{r\theta})(-\sin\theta) R^2 \sin\theta d\theta\, d\varphi$, and once again integrating over the sphere's surface we find

$$F_t = \int_0^{2\pi}\int_0^{\pi} \left(\tau_{r\theta}|_{r=R} \sin\theta \right) R^2 \sin\theta\, d\theta\, d\varphi \tag{117}$$

From Eq 110,

$$\tau_{r\theta}\Big|_{r=R} = \frac{3}{2}\frac{\eta V_\infty}{R}\sin\theta \tag{118}$$

which results in a net *frictional drag* from the tangential flow of

$$F_t = 4\pi\eta R V_\infty \tag{119}$$

Thus, the total force on our sphere in the flowing fluid is

$$F = \frac{4}{3}\pi R^3 \rho g + 6\pi\eta R V_\infty \tag{120}$$

The force has two terms, as expected: the first due to gravity (the weight of the fluid), exerted even if the fluid is stationary, and the second associated with fluid motion, sometimes called the "drag force." Both forces act in the same direction, opposing the direction of fluid flow. Sometimes, you will see the quantity $6\pi\eta R = b$ called the "Stokes radius," leading to a nice form of the force equation:

$$F = \frac{4}{3}\pi R^3 \rho g + b V_\infty = \frac{m_{\text{sphere}} g \rho}{\rho_{\text{sphere}}} + b V_\infty \tag{121}$$

Equation 120 is known as Stoke's law, and from it we may determine the terminal velocity of a falling sphere. Consider a sphere falling in a stagnant fluid of density ρ_s. In this case V_∞ is the *relative* velocity of the fluid with respect to the sphere, which in this case is just the velocity of the falling sphere since the fluid is stationary. The static and drag Stoke's forces act opposite the direction that the sphere falls, and at the terminal velocity V_t, precisely balance the sphere's weight. If the sphere has density ρ_s, this means

$$\frac{4}{3}\pi R^3 \rho g + 6\pi\eta R V_t = \frac{4}{3}\pi R^3 \rho_s g \tag{122}$$

This leads to a terminal velocity of

$$V_t = \frac{2gR^2\left(\rho_s - \rho\right)}{9\eta} \tag{123}$$

As expected, it is the relative density of fluid and solid that determine the behavior of the sphere: if the sphere is more dense than the fluid, it sinks, and if it is less dense than the fluid, it rises. If the particle radius and densities are known, a velocity measurement allows one to deduce the viscosity of a fluid. Conversely, if the densities and viscosity are known, one can find the radius of a small particle. The latter is useful in so-called *fluidized bed* particle separators, allowing one to separate particles by their size distribution.

9.3 Correction to Stoke's law for small velocities

One small detail remains: Stoke's law becomes inaccurate when the velocity of falling droplets is less than about 10^{-3} m/s. Droplets having this and smaller velocities have radii on the order of $2\,\mu$m for free-fall in air, comparable to the mean free path of air molecules, a condition which violates one of the assumptions made in deriving Stokes's law. We must add a correction to Stoke's law. As it turns out, the correction is straightforward: we need only replace the viscosity by an effective value, which includes a correction factor (the Cunningham factor):

$$\eta \longrightarrow \eta_{eff} = \frac{\eta}{1 + \frac{b}{Pr}} \tag{124}$$

Here P is the atmospheric pressure, r is the radius of the drop, and b is a constant factor.

Viscosity of dry air as a function of temperature

The viscosity of air can be computed using Sutherland's formula[xi]

$$\eta(T) = \eta_o \left(\frac{T}{T_o}\right)^{3/2} \frac{T_o + S}{T + S} \tag{125}$$

[xi] From http://www.epa.gov/EPA-AIR/2005/July/Day-13/a11534d.htm and http://en.wikipedia.org/wiki/Viscosity.

Here η is the viscosity in poise ($\mathrm{Ns/m^2}$) at the input temperature T, μ_o is the reference viscosity in poise ($\mathrm{Ns/m^2}$) at a reference temperature T_o, T is the input temperature in Kelvin, T_o is a reference temperature in Kelvin, and S is an effective temperature called Sutherland's constant. For dry air, $\eta_o = 1.716 \times 10^{-5}\,\mathrm{Ns/m^2}$ at $T_o = 273\,\mathrm{K}$, $S = 111\,\mathrm{K}$, valid over a temperature range of $0 - 555\,\mathrm{K}$. [xii]

[xii] Reference data: at 300 K, $\eta = 1.846 \times 10^{-5}\,\mathrm{Ns/m^2}$. Table A.4 in F.P. Incropera and D.P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 3rd ed, John Wiley, New York, NY, 1990. See also table B.4 in D.R. Poirier and G.H. Geiger, *Transport Phenomena in Materials Processing*, TMS, Warrendale, PA, 1994)