University of Alabama
Department of Physics and Astronomy

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# PH 253 Exam I Solutions

**1.** A train 1/2 km long (as measured by an observer on the train) is traveling at a speed of 100 km/hr. Two lightning bolts strike the ends of the train simultaneously as determined by an observer on the ground. What is the time separation as measured by an observer on the train?

***Solution:*** Here we have two events – lightning bolts striking two ends of a train – which are spatially separated. For this, we will want to use the Lorentz transformation for time intervals. The main trick to this problem is to keep straight which quantities are measured by which observer. Let the reference frame of the observer on the train be $O'$ and the reference frame of the observer on the ground be $O$. The train's length of 500 m is according to an observer at rest with respect to the train, not on the ground. The time delay according to the observer on the ground is zero, and we wish to find the time delay according to an observer on the train. We know then

$$\Delta x' = 500\,\text{m} \qquad \text{proper length of train}$$
$$\Delta t = 0 \qquad \text{time delay from ground}$$
$$v = 100\,\text{km/hr} \approx 27.8\,\text{m/s} \qquad \text{relative velocity of train}$$

The Lorentz transformation for time intervals reads

$$\Delta t = \gamma \left( \Delta t' - \frac{v \Delta x'}{c^2} \right) = 0 \tag{1}$$

Solving for $\Delta t'$,

$$0 = \gamma \left( \Delta t' - \frac{v \Delta x'}{c^2} \right) \tag{2}$$

$$\Delta t' = \frac{v \Delta x'}{c^2} \approx 1.5 \times 10^{-13}\,\text{s} \tag{3}$$

**2.** The speed of light in still water is $c/n$, where $n$ is the index of refraction, approximately $n = 4/3$ for water. Fizeau, in 1851, found that the speed (relative to the laboratory) of light in water moving at speed $V$ (relative to the laboratory) could be expressed as

$$u = \frac{c}{n} + kV \tag{4}$$

where the "dragging coefficient" was measured by him to be $k \approx 0.44$. Determine the value of $k$ predicted by the Lorentz velocity transformations. Note $(1+x)^{-1} \approx 1-x$ for $x \ll 1$.

***Solution:*** If the velocity of light in still water is $u'$ the velocity of light for an observer who views the water moving at velocity $V$ is found from the relativistic velocity addition formula:

$$u = \frac{u' + V}{1 + u'V/c^2} \tag{5}$$

From here we use $u' = c/n$ and approximate the denominator, since $V \ll c$:

$$u = \frac{u' + V}{1 + u'V/c^2} \approx \frac{\frac{c}{n} + V}{1 + \frac{V}{cn}} \approx \left(\frac{c}{n} + V\right)\left(1 - \frac{V}{cn}\right) = \frac{c}{n} - \frac{V}{n^2} + V - \frac{V^2}{cn} \tag{6}$$

We may neglect the last term with $c$ in the denominator if $V \ll c$, giving

$$u = \frac{c}{n} + V\left(1 - \frac{1}{n^2}\right) = \frac{c}{n} + kV \tag{7}$$

With $n = 4/3$, we find $k \approx 0.44$.

**3.** An electron initially moving at constant speed $v$ is brought to rest with uniform deceleration $a$ lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during this deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time $t$ and the classical electron radius $r_e = e^2/4\pi\epsilon_o mc^2$.

***Solution:*** The power emitted by a charge $e$ with acceleration $a$ is

$$P = \frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{8}$$

In this case, we know that $a = v/t$. The energy radiated in time $t$ is just $U = Pt$, so

$$U = Pt = \frac{e^2 v^2}{6\pi\epsilon_o c^3 t} \tag{9}$$

The ratio of this energy to the kinetic energy before deceleration is

$$\frac{U}{K} = \frac{1}{\frac{1}{2}mv^2} \frac{e^2 v^2}{6\pi\epsilon_o c^3 t} = \frac{e^2}{3\pi\epsilon_o mc^3 t} \tag{10}$$

Noting that the distance light travels in a time $t$ is $r_l = ct$ and using the expression for the classical electron radius above,

$$\frac{U}{K} = \frac{e^2}{3\pi\epsilon_o mc^3 t} = \frac{e^2}{4\pi\epsilon_o mc^2} \cdot \frac{4}{3} \cdot \frac{1}{ct} = \frac{4r_e}{3r_l} \tag{11}$$

**4.** In an experiment to find the value of $h$, light at wavelengths 218 and 431 nm were shone on a clean sodium surface. The potentials that stopped the fastest photoelectrons were 5.69 and 0.59 V, respectively. What values of $h$ and $W$, the sodium work function, are deduced?

***Solution:*** The photoelectric equation relates the incident light energy to the stopping potential required and the work function:

$$e\Delta V = hf - W = \frac{hc}{\lambda} - W \tag{12}$$

With the data given, you have two equations and two unknowns:

$$5.69\,\text{eV} = \left(\frac{3\times 10^8\text{m/s}}{218\times 10^{-9}\,\text{m}}\right) h - W \tag{13}$$

$$0.59\,\text{eV} = \left(\frac{3\times 10^8\text{m/s}}{431\times 10^{-9}\,\text{m}}\right) h - W \tag{14}$$

Solving, we find $W \approx 4.36\,\text{eV}$ and $h \approx 1.2\times 10^{-33}\,\text{J}\cdot\text{s}$. This is quite a bit higher than the work function of pure sodium (2.28 eV) and about twice as large as the accepted value of Planck's constant. Not a great experiment, we do much better in the PH255 lab.

**5.** In Compton scattering, an incident photon of energy $E_\gamma$ and momentum $p = hk$ scatters off of an electron at rest. The photon emerges at angle $\theta$ with reduced energy $E'_\gamma$ and momentum $p' = hk'$. The electron is ejected with energy $E_{e^-}$ and momentum $p_{e^-}$.

Show that the exiting photon's energy as a function of its energy and ejection angle $\theta$ is

$$E'_\gamma = \frac{mc^2}{(1-\cos\theta) + mc^2/E_\gamma} \tag{15}$$

***Solution:*** We need only the Compton equation and the energy of a photon.

$$\lambda' - \lambda = \frac{h}{mc}\left(1 - \cos\theta\right) \tag{16}$$
$$\frac{hc}{E'_\gamma} - \frac{hc}{E_\gamma} = \frac{h}{mc}\left(1 - \cos\theta\right) \tag{17}$$
$$\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{1}{mc^2}\left(1 - \cos\theta\right) \tag{18}$$
$$E'_\gamma = \frac{1}{\frac{1}{E_\gamma} + \left(1 - \cos\theta\right)} = \frac{mc^2}{\left(1 - \cos\theta\right) + mc^2/E_\gamma} \tag{19}$$

**6. (a)** An FM radio transmitter has a power output of 130 kW and operates at a frequency of 98.3 MHz. How many photons per second does the transmitter emit?

**(b)** A pulsed ruby laser emits light at 694.3 nm. For a 13.6 ps pulse containing 3.40 J of energy, how many photons are in the pulse? 1 ps is $10^{-12}$s.

***Solution:*** **(a)** We need for this only two things: the definition of power, and the fact that the energy emitted must come in the form of single photons. Since the frequency of the photons is fixed at $f = 98.3\,\text{MHz}$, then their energy is also fixed by $E = hf$. If the energy comes only in discrete bundles, one photon at a time, then the total energy is just the number of photons $N$, times the energy per photon $hf$. Power, you may recall, is just energy per unit time, and thus:

$$\mathscr{P} = \frac{\Delta E}{\Delta t} = \frac{\left(\text{number of photons}\right)\left(\text{energy per photon}\right)}{\Delta t} = \frac{Nhf}{\Delta t}$$

Now what we want is the number of photons per second, or $N/\Delta t$:

$$\frac{N}{\Delta t} = \frac{\mathscr{P}}{hf} = \frac{130 \times 10^3\,\text{W}}{\left(6.63 \times 10^{-34}\,\text{J}\cdot\text{s}\right)\left(98.3 \times 10^6\,\text{Hz}\right)} \approx 2 \times 10^{30}\,\text{photons/sec}$$

**(b)** The time duration of the pulse is irrelevant here - if we know the wavelength, we know the energy per photon. Since all of the photons have the same wavelength, if we also know the total energy of the pulse we know how many photons must make it up. The time would only be relevant if we were given the *power* of the pulse - *then* we would multiply power and time duration to get total energy. Since we already have energy, there is not much to do. Let $N$ be the number of photons, and $\lambda = 694.3\,\text{nm}$ their wavelength.

$$E = \left(\text{number of photons}\right)\left(\text{energy per photon}\right) = N\frac{hc}{\lambda} = 3.40\,\text{J}$$
$$\Longrightarrow \quad N = \frac{E\lambda}{hc} = \frac{\left(3.4\,\text{J}\right)\left(694.3 \times 10^{-9}\,\text{m}\right)}{\left(6.63 \times 10^{-34}\,\text{J}\cdot\text{s}\right)\left(3 \times 10^8\,\text{m/s}\right)} \approx 1 \times 10^{19}$$

Note that all the units cancel, and we end up with just a number, like we expect.

**7.** A molecule is known to exist in an unstable higher energy configuration for $\Delta t = 10\,\text{nsec}$, after which it relaxes to its lower energy stable state by emitting a photon.

**(a)** What uncertainty in the frequency $\Delta f$ of the emitted photon is implied? **(b)** If this state is being probed with Nuclear Magnetic Resonance (NMR) at a frequency of $f \approx 500\,\text{MHz}$, what is the relative uncertainty in the measurement, $\Delta f/f$?

***Solution:*** **(a)** The idea here is that our molecule has a stable ground state, whose energy we will call zero, and an excited state which has an energy $E_0$ higher. Thus, we can think of the molecule itself as being excited from an energy level zero to one $E_0$ higher. After excitation, the molecule can relax back to its lower energy state by emitting a photon of energy $E_0 = hf$ - if a photon carries away exactly this energy, the molecule is back to its lower state.

The excited state is unstable, and exists for only 10 ns. The energy-time uncertainty principle tells us is that states which exist for only a short time cannot have a definite energy - the shorter the lifetime of the excited state (or time an object spends in an energy level), the less well-defined the energy of that state is. In spectroscopy, this usually means that excited states with finite lifetime do not have definite energies, and what on would expect to be a narrow spectroscopic feature (a resonance line in this case) is actually broadened. For our molecule, if the lifetime of the excited state $\Delta t$ is short, the energy of the excited state $E_0$ is actually not definite, but has a “spread" or uncertainty of $\Delta E$. If this is the case, then the emitted photon must share the same uncertainty in *frequency*[i]: $\Delta E = h\Delta f$. Thus:

$$\begin{aligned}
\Delta E \Delta t = h\Delta f \Delta t &\geqslant \frac{\hbar}{2} \\
\implies \quad \Delta f &\geqslant \frac{\hbar}{2h\Delta t} \\
&\geqslant \frac{1}{4\pi\Delta t} = \frac{1}{4 \cdot \pi \cdot 10^{-8}\,\text{s}} \\
&\gtrsim 8\,\text{MHz}
\end{aligned}$$

In the second line, remember that $\hbar = h/2\pi$. Thus, our resonance line will have an intrinsic width of 8 MHz due purely to the uncertainty principle. If all extrinsic linewidths are minimized as far as possible, this sort of “lifetime broadening" is indeed observable for excited states that are sufficiently shortly-lived.

[i]Since Planck's constant is ... well ... *constant* the uncertainty must be in frequency, or equivalently, wavelength.

**(b)** The relative uncertainty, if we were measuring a NMR resonance line at 500 MHz[ii] is just the linewidth divided by the base frequency:

$$\frac{\Delta f}{f} = \frac{8\,\text{MHz}}{500\,\text{MHz}} \approx 1.6\%$$

A small effect, perhaps, but not at all unmeasurable.

[ii]This would require a magnetic field of about 11.75 T, which is accessible. NMR spectrometers running at 500 MHz are not uncommon.