Name & CWID

PH 253 Exam 2

Instructions

1. Solve four of the six problems below. All problems have equal weight.
2. Do your work on separate sheets.
3. Bring your exam paper with you when you leave - you need it for the next homework.
4. You are allowed 1 sheet of standard 8.5x11 in paper and a calculator.

LeClair
F10

1. Given the wave function

$$\psi(x) = \begin{cases} Ne^{\kappa x} & x < 0 \\ Ne^{-\kappa x} & x > 0 \end{cases} \tag{1}$$

(a) Find N needed to normalize ψ.
(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and Δx.

2. An electron in a hydrogen atom is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{3}\,(2a_o)^{3/2}} \frac{r}{a_o} e^{-r/2a_o} \tag{2}$$

where a_o is the Bohr radius.

(a) What is the *most probable* value of r?
(b) What is $\langle r \rangle$?

3. *Quantum harmonic oscillator.* The harmonic oscillator potential is $V(x) = \frac{1}{2}m\omega_o^2x^2$; a particle of mass m in this potential oscillates with frequency ω_o. The ground state wave function for a particle in the harmonic oscillator potential has the form

$$\psi(x) = Ae^{-ax^2} \tag{3}$$

(a) By substituting $V(x)$ and $\psi(x)$ into the one-dimensional time-independent Schrödinger equation, find expressions for the ground-state energy E and the constant a in terms of m, $\hbar$, and ω_o.
(b) Apply the normalization condition to determine the constant A in terms of m, $\hbar$, and ω_o.

4. By considering the visible spectrum of hydrogen and He^+, show how you could determine spectroscopically if a sample of hydrogen was contaminated with helium. (Hint: look for differences in the visible emission lines, $\lambda \approx 390 \sim 750$ nm. A difference of 10 nm is easily measured.)

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5. Find $\langle x \rangle$, $\langle x^2 \rangle$, and $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ (in terms of a) for a particle in the ground state of the one-dimensional simple harmonic oscillator, where:

$$\psi_0 = \sqrt{\frac{1}{a\sqrt{\pi}}} e^{-x^2/2a^2} \tag{4}$$

The following integrals may be useful:

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \qquad \int_{-\infty}^{\infty} x^3 e^{-ax^2}\,dx = \int_{-\infty}^{\infty} x e^{-ax^2}\,dx = 0 \qquad \int_0^\infty x^4 e^{-ax^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}} \tag{5}$$

6. A particle bound in a certain one-dimensional potential has a wave function described by the following equations:

$$\psi(x) = \begin{cases} 0 & x < -L \\ Ae^{-ikx} \cos\frac{\pi x}{L} & -L \leqslant x \leqslant L \\ 0 & x > L \end{cases} \tag{6}$$

(a) Find the value of the normalization constant A by enforcing the condition $\int_{\text{all } x} |\psi(x)|^2\,dx = 1$.
(b) What is the probability that the particle will be found between $x=0$ and $x=L/4$?

Constants:

$$N_A = 6.022 \times 10^{23} \text{ things/mol}$$
$$k_e \equiv 1/4\pi\epsilon_o = 8.98755 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}$$
$$\epsilon_o = 8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$$
$$\mu_o \equiv 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$$
$$e = 1.60218 \times 10^{-19} \text{ C}$$
$$h = 6.6261 \times 10^{-34} \text{ J} \cdot \text{s} = 4.1357 \times 10^{-15} \text{ eV} \cdot \text{s}$$
$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84 \text{ eV} \cdot \text{nm}$$
$$k_B = 1.38065 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 8.6173 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$$
$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792 \times 10^8 \text{ m/s}$$
$$m_e = 9.10938 \times 10^{-31} \text{ kg} \qquad m_e c^2 = 510.998 \text{ keV}$$
$$m_p = 1.67262 \times 10^{-27} \text{ kg} \qquad m_p c^2 = 938.272 \text{ MeV}$$
$$m_n = 1.67493 \times 10^{-27} \text{ kg} \qquad m_n c^2 = 939.565 \text{ MeV}$$

Schrödinger

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi + V(x)\Psi \quad \text{1D time-dep}$$
$$E\psi = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + V(x)\psi \quad \text{1D time-indep}$$
$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 \qquad P(\text{in } [x, x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$
$$\int_0^{\infty} |\psi(r)|^2 \, 4\pi r^2 \, dr = 1 \qquad P(\text{in } [r, r+dr]) = 4\pi r^2 |\psi(r)|^2 \qquad \text{3D}$$
$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x) \, dx \qquad \text{1D} \qquad \langle r^n \rangle = \int_0^{\infty} r^n P(r) \, dr \qquad \text{3D}$$
$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Basic Equations:

$$\vec{F}_{net} = m\vec{a} \text{ Newton's Second Law}$$
$$\vec{F}_{\text{centr}} = -\frac{mv^2}{r} \hat{r} \text{ Centripetal}$$
$$\vec{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} = q_2 \vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$
$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e \frac{q_1}{r_{12}^2} \hat{r}_{12}$$
$$\vec{F}_B = q\vec{v} \times \vec{B}$$
$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Oscillators

$$E = \left(n + \frac{1}{2}\right) hf$$
$$E = \frac{1}{2} kA^2 = \frac{1}{2} \omega^2 m A^2 = 2\pi^2 m f^2 A^2$$
$$\omega = 2\pi f = \sqrt{k/m}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2} n(n+1) x^2 \qquad \tan x \approx x + \frac{1}{3} x^3$$
$$e^x \approx 1 + x + \frac{1}{2} x \qquad \sin x \approx x - \frac{1}{6} x^3 \qquad \cos x \approx 1 - \frac{1}{2} x^2$$

Misc Quantum

$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$
$$\lambda_f - \lambda_i = \frac{h}{m_e c} (1 - \cos\theta)$$
$$\lambda = \frac{h}{|\vec{p}|} = \frac{h}{\gamma m v} \approx \frac{h}{mv}$$
$$\Delta x \Delta p \geqslant \frac{h}{4\pi} \qquad \Delta E \Delta t \geqslant \frac{h}{4\pi}$$
$$eV_{\text{stopping}} = KE_{\text{electron}} = hf - \varphi = hf - W$$

Bohr

$$E_n = -13.6 \text{ eV}/n^2 \qquad \text{Hydrogen}$$
$$E_n = -13.6 \text{ eV} \left(Z^2/n^2\right) \qquad Z \text{ protons, } 1\ e^-$$
$$E_i - E_f = -13.6 \text{ eV} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = hf$$
$$L = mvr = n\hbar$$
$$v^2 = \frac{n^2 \hbar^2}{m_e^2 r^2} = \frac{k_e e^2}{m_e r}$$

Quantum Numbers

$$l = 0, 1, 2, \ldots, (n-1) \qquad L^2 = l(l+1)\hbar^2$$
$$m_l = -l, (-l+1), \ldots, l \qquad L_z = m_l \hbar$$
$$m_s = -\pm \frac{1}{2} \qquad S_z = m_s \hbar \qquad S^2 = s(s+1)\hbar^2$$
$$\text{dipole transitions: } \Delta l = \pm 1, \Delta m_l = 0, \pm 1, \Delta m_s = 0$$
$$\mu_{sz} = \pm \mu_B$$
$$\vec{\mu}_s = 2\vec{S} \mu_B$$
$$E_\mu = -\vec{\mu} \cdot \vec{B}$$
$$J^2 = j(j+1)\hbar^2 \qquad j = l \pm \frac{1}{2}$$
$$J_z = m_j \hbar \qquad m_j = -j, (-j+1), \ldots, j$$

Calculus of possible utility:

$$\int \frac{1}{x} dx = \ln x + c$$
$$\int u \, dv = uv - \int v \, du$$
$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$
$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C$$
$$\frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x}$$
$$\int e^{-ax} \, dx = -\frac{1}{a} e^{-ax} + C$$
$$\int_0^{\infty} x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}$$
$$\int_0^{\infty} x^2 e^{-ax^2} \, dx = \frac{1}{4} \sqrt{\frac{\pi}{a^3}}$$
$$\int_{-\infty}^{\infty} x^3 e^{-ax^2} \, dx = \int_{-\infty}^{\infty} x e^{-ax^2} \, dx = 0$$
$$\int_0^{\infty} x^4 e^{-ax^2} \, dx = \frac{3}{8} \sqrt{\frac{\pi}{a^5}}$$