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PH 253 Exam II

Instructions

1. Solve three of the six problems below. All problems have equal weight.
2. Clearly mark your which problems you have chosen.
3. Do your work on separate sheets. Staple them to this exam paper when you are finished.
4. You are allowed 1 sheet of standard 8.5x11 in paper and a calculator.

1. The state of a free particle is described by the following wave function

$$\psi(x) = \begin{cases} 0 & x < -b \\ A & -b \leqslant x \leqslant 2b \\ 0 & x > 2b \end{cases} \tag{1}$$

(a) Determine the normalization constant A.
(b) What is the probability of finding the particle in the interval $[0, b]$?
(c) Determine $\langle x \rangle$ and $\langle x^2 \rangle$ for this state.
(d) Find the uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

2. The Schrödinger equation for a simple harmonic oscillator of mass m can be written

$$-a^4 \frac{d^2\psi}{dx^2} + x^2\psi = \frac{2E}{C}\psi \tag{2}$$

where $a^4 = \hbar^2/mC$, C is the force constant, and E the energy. [i]
(a) Below are the wave functions for the first two states; find their energies in terms of $\hbar\omega_o$.
(b) Suggest a general formula for energy the n^{th} state. How does it differ from Planck's hypothesis for the energy of his oscillators?

$$\psi_0 = \left(\frac{1}{a\sqrt{\pi}}\right)^{1/2} e^{-x^2/2a^2}$$
$$\psi_1 = \left(\frac{1}{2a\sqrt{\pi}}\right)^{1/2} 2\left(\frac{x}{a}\right) e^{-x^2/2a^2}$$

To save you some time, we note $\frac{d}{dx}(e^{-x^2/2a^2}) = -\frac{x}{a^2}e^{-x^2/2a^2}$ and $\frac{d^2}{dx^2}(e^{-x^2/2a^2}) = \frac{x^2-a^2}{a^4}e^{-x^2/2a^2}$

3. A phenomenological expression for the potential energy of a bond as a function of spacing is given by

$$U(r) = \frac{A}{r^n} - \frac{B}{r^m} \tag{3}$$

For a stable bond, $m < n$. Show that the molecule will break up when the atoms are pulled apart to a distance

[i] Note $\omega_o = 2\pi f_o = \sqrt{C/m}$, $a = (\hbar/\sqrt{mC})^{1/2} = \sqrt{\hbar/m\omega_o}$.

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$$r_b = \left(\frac{n+1}{m+1}\right)^{1/(n-m)} r_o \tag{4}$$

where r_o is the equilibrium spacing between the atoms. Be sure to note your criteria for breaking used to derive the above result.

4. (a) Using the Bohr model, what wavelength of photon is emitted when an electron in a hydrogen atom makes a transition from the $4f$ to $3d$ state?
(b) Show that in the presence of a magnetic field, the $4f \to 3d$ transition in hydrogen appears as three spectral lines. You may ignore spin, and assume only dipole transitions will occur (see formula sheet).

5. By considering the visible spectrum of hydrogen and He^+, show how you could determine spectroscopically if a sample of hydrogen was contaminated with helium. (Hint: look for differences in the visible emission lines, $\lambda \approx 390 \sim 750\,\mathrm{nm}$. A difference of $10\,\mathrm{nm}$ is easily measured.)

6. Find the most probable radius and the expected value of the radial position $\langle r \rangle$ of an electron in the $2s$ state.

$$\psi_{2s} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_o}\right)^{3/2}\left(2 - \frac{r}{a_o}\right)e^{-r/2a_o} \tag{5}$$

where a_o is the Bohr radius, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.529 \times 10^{-10}\,\mathrm{m}$. Make use of the integrals given on the formula sheet.

Constants:

$$N_A = 6.022\times10^{23}\ \text{things/mol}$$
$$k_e \equiv 1/4\pi\epsilon_o = 8.98755\times10^{9}\ \mathrm{N\cdot m^2\cdot C^{-2}}$$
$$\epsilon_o = 8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$$
$$\mu_o \equiv 4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$$
$$e = 1.60218\times10^{-19}\ \mathrm{C}$$
$$h = 6.6261\times10^{-34}\ \mathrm{J\cdot s} = 4.1357\times10^{-15}\ \mathrm{eV\cdot s}$$
$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84\ \mathrm{eV\cdot nm}$$
$$k_B = 1.38065\times10^{-23}\ \mathrm{J\cdot K^{-1}} = 8.6173\times10^{-5}\ \mathrm{eV\cdot K^{-1}}$$
$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.99792\times10^{8}\ \mathrm{m/s}$$
$$m_e = 9.10938\times10^{-31}\ \mathrm{kg} \qquad m_e c^2 = 510.998\ \mathrm{keV}$$
$$m_p = 1.67262\times10^{-27}\ \mathrm{kg} \qquad m_p c^2 = 938.272\ \mathrm{MeV}$$
$$m_n = 1.67493\times10^{-27}\ \mathrm{kg} \qquad m_n c^2 = 939.565\ \mathrm{MeV}$$

Schrödinger

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi$$
$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi$$
$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1 \qquad P(\text{in }[x,x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$
$$\int_{0}^{\infty}|\psi(r)|^2\,4\pi r^2\,dr = 1 \qquad P(\text{in }[r,r+dr]) = 4\pi r^2|\psi(r)|^2 \qquad \text{3D}$$
$$\langle x^n\rangle = \int_{-\infty}^{\infty}x^n P(x)\,dx \qquad \text{1D} \qquad \langle r^n\rangle = \int_0^{\infty} r^n P(r)\,dr \qquad \text{3D}$$
$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Basic Equations:

$$\vec{F}_{net} = m\vec{a}\ \text{ Newton's Second Law}$$
$$\vec{F}_{centr} = -\frac{mv^2}{r}\,\hat{r}\ \text{ Centripetal}$$
$$\vec{F}_{12} = k_e\frac{q_1q_2}{r_{12}^2}\,\hat{r}_{12} = q_2\vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$
$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{r}_{12}$$
$$\vec{F}_B = q\vec{v}\times\vec{B}$$
$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Oscillators

$$E = \left(n+\frac{1}{2}\right)hf$$
$$E = \frac{1}{2}kA^2 = \frac{1}{2}\omega^2 mA^2 = 2\pi^2 mf^2A^2$$
$$\omega = 2\pi f = \sqrt{k/m}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3$$
$$e^x \approx 1 + x + \frac{1}{2}x \qquad \sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

Misc Quantum

$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$
$$\lambda_f - \lambda_i = \frac{h}{m_e c}(1-\cos\theta)$$
$$\lambda = \frac{h}{|\vec{p}|} = \frac{h}{\gamma mv} \approx \frac{h}{mv}$$
$$\Delta x\Delta p \geqslant \frac{h}{4\pi} \qquad \Delta E\Delta t \geqslant \frac{h}{4\pi}$$
$$eV_{\text{stopping}} = KE_{\text{electron}} = hf - \varphi = hf - W$$

Bohr

$$E_n = -13.6\ \mathrm{eV}/n^2 \qquad \text{Hydrogen}$$
$$E_n = -13.6\ \mathrm{eV}\left(Z^2/n^2\right) \qquad Z\text{ protons, 1 }e^-$$
$$E_i - E_f = -13.6\ \mathrm{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = hf$$
$$L = mvr = n\hbar$$
$$v^2 = \frac{n^2\hbar^2}{m_e^2r^2} = \frac{k_e e^2}{m_e r}$$

Quantum Numbers

$$l = 0,1,2,\ldots,(n-1) \qquad L^2 = l(l+1)\hbar^2$$
$$m_l = -l,(-l+1),\ldots,l \qquad L_z = m_l\hbar$$
$$m_s = -\pm\frac{1}{2} \qquad S_z = m_s\hbar \qquad S^2 = s(s+1)\hbar^2$$
$$\text{dipole transitions: } \Delta l = \pm1, \Delta m_l = 0,\pm1, \Delta m_s = 0$$
$$\mu_{sz} = \pm\mu_B$$
$$\vec{\mu}_s = 2\vec{S}\mu_B$$
$$E_\mu = -\vec{\mu}\cdot\vec{B}$$
$$J^2 = j(j+1)\hbar^2 \qquad j = l\pm\frac{1}{2}$$
$$J_z = m_j\hbar \qquad m_j = -j,(-j+1),\ldots,j$$

Calculus of possible utility:

$$\int\frac{1}{x}\,dx = \ln x + c$$
$$\int u\,dv = uv - \int v\,du$$
$$\int\sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int\cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$
$$\int_0^{\infty}x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^{\infty}x^2e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$
$$\int_{-\infty}^{\infty}x^3e^{-ax^2}\,dx = \int_{-\infty}^{\infty}xe^{-ax^2}\,dx = 0$$
$$\int_0^{\infty}x^4e^{-ax^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$