UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

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Final Exam Sample Problems

1. The orbital speed of the Earth around the Sun is 30 km/s. In one year, how many seconds do the clocks on the Earth lose with respect to the clocks of an inertial reference frame at rest relative to the Sun? *Hint: if* v/c *is small, the following approximation is valid:*

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1-\frac{1}{2}\frac{v^2}{c^2}$$

If we are to consider an inertial frame at rest relative to the sun, it must be the one keeping the proper time interval Δt_p. Over the course of one year, we have

$$\Delta t_p = 1\,\text{yr} \approx 3.156\times10^7\,\text{s} \tag{1}$$

The observers on earth are in motion relative to the sun, and therefore they measure a dilated time interval $\Delta t' = \gamma\Delta t_p$. We are asked for the *difference* between the two clocks after one year as measured in the sun's inertial frame, or

$$\text{difference} = \Delta t' - \Delta t_p = \gamma\Delta t_p - \Delta t_p = (\gamma-1)\,\Delta t_p \tag{2}$$

Since in this case the relative velocity of earth is small with respect to c, $v = 30\,\text{km/s} = 3\times10^4\,\text{s}$ so $v/c = 10^{-4}$, we can use the second approximation given.

$$\gamma - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \approx 1 + \frac{1}{2}\frac{v^2}{c^2} - 1 = \frac{1}{2}\frac{v^2}{c^2} \tag{3}$$

Thus,

$$\text{difference} \approx \frac{1}{2}\frac{v^2}{c^2}\Delta t_p = \frac{1}{2}\left(10^{-4}\right)^2\left(3.156\times10^7\,\text{s}\right) = \left(5\times10^{-9}\right)\left(3.156\times10^7\,\text{s}\right) \approx 0.16\,\text{s} \tag{4}$$

2. A cannonball flies through our classroom at a speed of $0.30c$. Measurement of the transverse diameter ("width") of the cannonball gives a result of 0.20 m. What can you predict for the measurement of the longitudinal diameter ("length") of the cannonball?

If the cannonball is moving at high speed relative to an observer, along the direction of motion its length will appear *shorter* by a factor γ. Along the transverse direction (perpendicular to the direction of motion), no contraction will occur and the width observed will be the proper one $L_p = 0.2\,\mathrm{m}$. Given the cannonball's speed of $v=0.3c$, the length will appear to be

$$L' = \frac{L_p}{\gamma} = (0.2\,\mathrm{m})\sqrt{1-\frac{v^2}{c^2}} \approx 0.19\,\mathrm{m}$$

3. Consider a particle of mass m moving at a speed of $0.10c$. What is its kinetic energy according to the relativistic formula? What is its kinetic energy according to the Newtonian formula? What is the percent deviation between these two results?

The classical formula for kinetic energy gives:

$$K_c = \frac{1}{2}mv^2 = \frac{1}{2}m\,(0.10c)^2 = 0.005mc^2 \tag{5}$$

The relativistic formula yields – being *very* careful to carry enough digits – something just a bit larger:

$$K_r = (\gamma-1)\,mc^2 = \left(\frac{1}{\sqrt{1-0.01}} - 1\right)mc^2 \approx (1.00504-1)\,mc^2 = 0.00504mc^2 \tag{6}$$

The percent deviation is then (note that the mc^2 bits cancel)

$$\%\text{ deviation} = 100\% \times \left[\frac{K_r - K_c}{K_r}\right] = 100\% \times \left[\frac{0.00504mc^2 - 0.005mc^2}{0.00504mc^2}\right] \approx 0.8\% \tag{7}$$

Even at a tenth the speed of light, it is not a big difference.

4. **10 points.** A radioactive atom in a beam produced by an accelerator has a speed of $0.80c$ relative to the laboratory. The atom decays and ejects an electron of speed $0.50c$ relative to itself. What is the speed of the electron relative to the laboratory if ejected in **(a)** the forward direction? **(a)** The backward direction?

The essence of this problem is that we have one object, a radioactive atom, going at $0.80c$ relative to the laboratory, and it ejects a second object, an electron, at $0.50c$ relative to itself. How fast is the second object going with respect to the lab? This is just velocity addition, the same way we would find out without relativity - we add the first object's velocity to the second.

First, let's be clear on our definitions. Let the laboratory reference frame be an unprimed system, with the the atom moving along the $+x$ direction. Thus, if the electron is ejected in the forward direction, its velocity is positive, if it is ejected in the backward direction, its velocity is positive. Let the reference frame of the first object (the radioactive atom) be the primed system. Thus,

$$v_1 = \text{velocity of the atom, relative to the lab}$$
$$v_2 = \text{velocity of the electron, relative to the lab}$$
$$v_2' = \text{velocity of the electron, relative to the atom}$$

The velocity of the electron relative to the lab frame is just the velocity of the atom relative to the lab, plus the velocity of the electron relative to the atom, corrected by our relativistic factor. For a forward-ejected electron, the latter velocity is positive, $v_2' = 0.5c$, and we need just apply the velocity addition formula:

$$\text{forward ejection:} \quad v_2 = \frac{v_1 + v_2'}{1 + \frac{v_1 v_2'}{c^2}} = \frac{0.8c + 0.5c}{1 + \frac{(0.5c)(0.8c)}{c^2}} \approx 0.93c$$

When the electron is ejected backward, then $v_2' = -0.5c$, but the rest is the same:

$$\text{backward ejection:} \quad v_2 = \frac{v_1 + v_2'}{1 + \frac{v_1 v_2'}{c^2}} = \frac{0.8c + (-0.5c)}{1 + \frac{(-0.5c)(0.8c)}{c^2}} \approx 0.5c$$

A nice numerical coincidence! In the second case, the numbers were 'doctored' to make the velocity come out the same in either reference frame, except for the change of sign. It is nothing more than a coincidence though.

5. **20 points.** A muon formed high in the Earth's atmosphere travels at $v = 0.990c$ for $4.60\,\text{km}$ before it decays into an electron, a neutrino, and an antineutrino ($\mu^- \rightarrow e^- + \nu + \overline{\nu}$). **(a)** How long does the muon live, as measured in its own reference frame? **(b)** How far does the Earth travel, as measured in the frame of the muon?

Let the earth be in reference frame O' (primed frame), and the muon itself in O (unprimed frame). First, since we know we will need it, for $v = 0.990c$, $\gamma = 7.09$. Next, the numbers we are given are measured in the earth's reference frame, so it will be easiest to calculate the time in the earth's frame first. The muon, according to earthbound observers, travels $4600\,\text{m}$ at a speed of $0.990c$, so the apparent decay time is just distance divided by velocity.

$$\Delta t'_{\text{earth}} = \frac{4600\,\text{m}}{0.990\,(3 \times 10^8\,\text{m/s})} \approx 1.54 \times 10^{-5}\,\text{s} = 15.4\,\mu\text{s}$$

This is *not* the proper time - proper time is measured in the muon's own frame. According to the muon, the earth is moving toward them! Given γ and time measured on earth, we can find the proper time in the muon's frame easily:

$$\Delta t_p = \frac{\Delta t'_{earth}}{\gamma} \approx \frac{1.54\,\mu s}{7.09} = 2.18\,\mu s$$

This makes sense - since the people on earth are the moving observers in this case, they should see a longer time interval. About seven times longer, in this case, since $\gamma \approx 7$. The muon is at rest in its own frame, and measures the shorter proper time interval. Now we have the proper time, meaured in the muon's reference frame, and the relative velocity, so we can calculate the distance from the muon's point of view using quantities valid in its reference frame.

$$d_\mu = v\Delta t_p = v\frac{\Delta t'_{earth}}{\gamma} \approx 649\,m$$

6. A plane, 300 MHz electromagnetic wave is incident normally on a surface of area $50\,cm^2$. If the intensity of the wave is $9 \times 10^{-5}\,W/m^2$, determine the rate at which photons strike the surface.

The quoted intensity is energy per unit time per unit surface area, and the product of intensity and surface area gives energy per unit time, or power:

$$\mathscr{P} = IA = \left(9 \times 10^5\,[W/m^2]\right)\left(5 \times 10^{-3}\,[m^2]\right) = 4.5 \times 10^{-7}\,[W] = \frac{\Delta E}{\Delta t} \tag{8}$$

The energy per unit time delivered by monochromatic photons[i] is just the number of photons per unit time multiplied by the energy per photon:

$$\mathscr{P} = hf\frac{\Delta N}{\Delta t} = 1.988 \times 10^{-25}\,[J]\frac{\Delta N}{\Delta t} \tag{9}$$

Equating our two expressions for power and solving for the number of photons per unit time,

$$\frac{\Delta N}{\Delta t} = \frac{IA}{hf} = \frac{4.5 \times 10^{-7}\,[J/s]}{1.988 \times 10^{-25}\,[J]} = 2.26 \times 10^{18}\,[s^{-1}] \tag{10}$$

7. The kinetic energies of photoelectrons range from zero to $4.0 \times 10^{-19}\,J$ when light of wavelength 300 nm falls on a surface. What is the stopping potential for this light? The threshold wavelength for this material?

[i] I.e., photons all of the same frequency/wavelength.

Typo in original problem: should be 4.0×10^{-19} J not 4.0×10^{-9} J.

The stopping potential is just the means by which the maximum kinetic energy is measured. Electrons are ejected from a metal surface with kinetic energy K, and a potential is applied between the metal and a separate electrode. When this potential is sufficient to prevent any electrons from reaching the electrode, the electrical potential energy required to reach the electrode is equal to the maximum electron kinetic energy. Thus, an energy balance gives us the stopping potential:

$$K_{max} = 4.0 \times 10^{-19}\,[\mathrm{J}] \left(\frac{1\,[\mathrm{eV}]}{1.6 \times 10^{-19}\,[\mathrm{J}]} \right) = 2.5\,\mathrm{eV} \tag{11}$$

$$eV_{stop} = K_{max} \quad \Longrightarrow \quad V_{stop} = 2.5\,[\mathrm{V}] \tag{12}$$

8. Prove that the photoelectric effect cannot occur for free electrons. Hint: try to conserve energy and momentum.

This is equivalent to HW3, problem 2. A free electron absorbing a photon cannot conserve both energy and momentum; a host crystal for the electron is required to take up some of the recoil momentum (it absorbs a negligible amount of the energy).

9. A 0.3 MeV X-ray photon makes a "head on" collision with an electron initially at rest. Using conservation of energy and momentum, find the recoil velocity of the electron. Check your result with the Compton formula.

Let E and E' be the initial and final energies of the photon, respectively. Conservation of energy then gives:

$$E = E' + (\gamma - 1)\,mc^2 \tag{13}$$

For a head-on collision, the photon will recoil in the opposite direction, and the electron along the photon's original direction. Conservation of momentum then yields

$$\frac{E}{c} = -\frac{E'}{c} + \gamma m v \tag{14}$$

Given that E, m, and c are known quantities, simultaneous solution of the two equations above to eliminate E' gives $v = 0.65c$.

10. Show that a free electron at rest cannot absorb a photon.

This is *also* equivalent to HW3, problem 2. A free electron absorbing a photon cannot conserve both energy and momentum; a host crystal for the electron is required to take up some of the recoil momentum.

11. In Compton scattering, what is the kinetic energy of the electron scattered at an angle φ to the incident photon?

See HW3 problem 9, where we derived the energy of the scattered electron as a function of ϕ and incident photon energy.

12. If the maximum energy imparted to an electron in Compton scattering is $45\,\text{keV}$, what is the wavelength of the incident photon?

Also see HW3. Numerical answer: $9.39 \times 10^{-3}\,\text{nm}$.

13. Determine the accelerating potential necessary to give an electron a de Broglie wavelength of $0.1\,\text{nm}$, which is the size of the interatomic spacing in a crystal.

From conservation of energy (neglecting relativistic effects),

$$eV = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{h}{\lambda}\right)^2 \tag{15}$$

$$V = \frac{h^2}{2me\lambda^2} \approx 151\,[\text{V}] \tag{16}$$

14. Determine the phase velocity of the wave corresponding to a de Broglie wavelength of $\lambda = h/p = h/mv$.

Phase velocity is nothing more than wavelength divided by period, or wavelength times frequency. The de wavelength can be written in terms of momentum, and the frequency can separately be found from Planck's energy hypothesis:

$$\lambda = \frac{h}{mv} \tag{17}$$

$$E = hf = mc^2 \tag{18}$$

$$v_p = \lambda f = \frac{h}{mv}\frac{mc^2}{h} = \frac{c^2}{v} \tag{19}$$

There are other was to go about this. For instance, using the angular frequency and wavevector, $\lambda f = \omega/k$, and thus

$$v_p = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v^2} \tag{20}$$

15. At what energy will the nonrelativistic calculation of the de Broglie wavelength of an electron be off by 5%?

In either case,

$$\lambda = \frac{h}{p} \tag{21}$$

In the relativistic case,

$$\left(K + mc^2\right)^2 = (pc)^2 + \left(mc^2\right)^2 \tag{22}$$

$$\text{or} \qquad p = \sqrt{2mK\left(1 + \frac{K}{2mc^2}\right)} \tag{23}$$

And thus

$$\lambda_{\text{rel}} = \frac{hc}{pc} = \frac{hc}{\sqrt{2mK\left(1 + \frac{K}{2mc^2}\right)}} \tag{24}$$

Classically, using $p^2/2m = K$,

$$\lambda_{\text{cl}} = \frac{h}{p} = \frac{h}{\sqrt{2mK}} \tag{25}$$

A 5% error means $\lambda_{\text{cl}}/\lambda_{\text{rel}} = 1.05$:

$$\frac{\lambda_{\text{class}}}{\lambda_{\text{rel}}} = \frac{h}{\sqrt{2mK}} \frac{\sqrt{2mK\left(1 + \frac{K}{2mc^2}\right)}}{h} = \sqrt{1 + \frac{K}{2mc^2}} \tag{26}$$

Given $mc^2 = 0.511\,\text{MeV}$ for an electron, solving yields $K = 0.105\,\text{MeV}$.

16. What is the uncertainty in the location of a photon of wavelength 300 nm if this wavelength is known to an accuracy of one part in a million?

The momentum of the photon is

$$p = \frac{h}{\lambda} \tag{27}$$

The uncertainty in the photon momentum can be related to the uncertainty in wavelength (as we have done before with wavelength and energy uncertainty).

$$\Delta p = \left|\frac{\partial p}{\partial \lambda}\right| \Delta\lambda = \frac{h}{\lambda^2}\Delta\lambda = \frac{p\Delta\lambda}{\lambda} \tag{28}$$

Using the uncertainty principle,

$$\Delta x \geqslant \frac{\hbar}{2\Delta p} = \frac{\hbar\lambda}{2p\Delta\lambda} \approx 23.9\,\mathrm{mm} \tag{29}$$

This seems like a lot, but keep in mind that a photon takes only about 80 ps[ii] to cover this distance!

17. What is the minimum uncertainty in the energy state of an atom if an electron remains in this state for 10^{-8} s?

We need only the energy-time uncertainty principle:

$$\Delta E \geqslant \frac{\hbar}{2\Delta t} \approx 3.29 \times 10^{-8}\,\mathrm{eV} \tag{30}$$

Again, this may not seem like a lot, but this is uncertainty corresponds to the energy of a 8 MHz photon in the microwave/radio range. It is also in the right ballpark for resonance frequencies studied in NMR (or MRI), and this sort of energy uncertainty is known as 'lifetime broadening' of the resonance peak.

18. Suppose the uncertainty in the momentum of a particle is equal to the particle's momentum. How is the minimum uncertainty in the particle's location related to is de Broglie wavelength?

We are given $\Delta p = p$, so

$$\Delta x \geqslant \frac{h}{4\pi\Delta p} = \frac{h}{4\pi p} = \frac{\lambda}{4\pi} \tag{31}$$

19. A particle of mass m is confined to a one-dimensional line of length L. From arguments based on the uncertainty principle, estimate the value of the smallest energy that the body can have.

The particle must be somewhere within the given segment, so the uncertainty in its position cannot be greater than L. If we say $\Delta x = L$, the uncertainty principle implies a momentum uncertainty of $\Delta p \geqslant h/4\pi L$. With maximum position uncertainty, we have a minimum momentum uncertainty. This in turn implies a minimum energy, since $K = p^2/2m$.

[ii] p = pico = 10^{-12}.

If we brazenly assume that the minimum momentum is just half of the uncertainty (i.e., the momentum may be zero plus or minus half the uncertainty), then $p_{\text{min}} = \frac{1}{2}\Delta p = h/8\pi L$. Thus,

$$K_{\text{min}} = \frac{p^2}{2m} = \frac{h^2}{128\pi^2 m L^2} \tag{32}$$

Given how crude our arguments are, the dependence on mass and length agrees reasonably well with the minimum energy for a particle in a one-dimensional potential well,

$$E_1 = \frac{h^2}{8mL^2} \tag{33}$$

20. An electron makes a transition from the $n = 5$ to the $n = 2$ state in hydrogen. Within the Bohr model, find the energy and wavelength of the emitted photon.

434 nm, 2.86 eV.

21. How many different photons can be emitted by hydrogen atoms that undergo transitions to the ground state from the $n=5$ state?

One can brute-force this quickly enough to find that there are 10 transitions. One may also solve the problem for an arbitrary n. More generally, the number of possible transitions is just equal to the number of ways one can choose 2 numbers from a set of n without worrying about their order (i.e., the number of combinations choosing 2 elements from a set of n):

$$(\text{number of different photons}) = \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\,(n-1)}{2} \tag{34}$$

This works because the order does not matter: if we have $n=5$ and pick the pair $(4, 3)$ or $(3, 4)$ we need only count the first ordering, not the second. Hence, we use a combination rather than a permutation. Further, you can easily convince yourself that this includes all possible intermediate states, accounting for multi-step transitions such as $5 \rightarrow 3 \rightarrow 1$. Given $n=5$, we readily find 10 different transitions from the formula above.

22. For hydrogen within the Bohr model, show that when $n \gg 2$ the frequency of the emitted photon in a transition from n to $n-1$ equals the rotational frequency, in agreement with the correspondence principle.

The rotational frequency – revolutions per second – is just the angular frequency in radians per second divided by the 2π radians in one full rotation. This is easily found from the orbital velocity in the Bohr model:

$$\frac{\omega_n}{2\pi} = \frac{v_n}{2\pi r_n} = \frac{2\pi k_e e^2/nh}{2\pi n^2 h^2/4\pi^2 k_e m e^2} = \frac{4\pi^2 k_e^2 m e^4}{n^3 h^3} \tag{35}$$

The frequency of the emitted photon is

$$f = \frac{c}{\lambda} = cR_\infty \left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] = cR_\infty \frac{2n-1}{n^2 (n-1)^2} \tag{36}$$

Here $R_\infty = 2\pi^2 k^2 e^4 m/h^3 c$ is the Rydberg constant. For $n \gg 1$, $n \approx n-1$, and

$$f \approx cR_\infty \frac{2n}{n^2 n^2} = \frac{4\pi^2 k^2 m e^4}{n^3 h^3} \tag{37}$$

which is the same as the rotational frequency given above. This is an illustration of Bohr's "correspondence principle," which stats that for large n a quantum equation should give the same result as the corresponding classical equation. From classical theory, the radiation emitted should have the a frequency equal to the rotational frequency.