PH 253 Final Exam: Solution

1. A particle of mass m is confined to a one-dimensional box of width L, that is, the potential energy of the particle is infinite everywhere except in the interval $0<x<L$, where its potential energy is zero. The particle is in its ground state. What is the probability that a measurement of the particle's position will yield a result in the left quarter of the box? The wavefunction for a particle in a 1D box may be written

$$\psi(x) = A \sin Bnx \tag{1}$$

where A and B are constants you will need to find, and n is an integer. *Hint: normalize and apply boundary conditions.*

Solution: *UA physics graduate qualifying exam, 2002.* Our boundary conditions are that the wavefunction vanish at the boundaries of the box $x=0$ and $x=L$, since the potential is infinite outside of that region.[i] This allows us to determine B already:

$$\psi(0) = A \sin 0 = 0 \tag{2}$$

$$\psi(L) = A \sin BnL = 0 \qquad \Longrightarrow \qquad BnL = n\pi \qquad \Longrightarrow \qquad B = \frac{\pi}{L} \tag{3}$$

Thus, $\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$. We need only determine the overall constant A, which can be done by enforcing normalization (i.e., the probability density integrated over all space must give unity). Since the wavefunction vanishes outside $[0, L]$, we need only integrate over that interval.

$$1 = \int_0^L |\psi(x)|^2 \, dx = \int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}A^2 L \qquad \Longrightarrow \qquad A = \sqrt{\frac{2}{L}} \tag{4}$$

Thus,

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \tag{5}$$

The probability that the box will be found in the left quarter of the box is determined by integrating the probability density over that interval:

[i] We also require that the derivative of the wavefunction vanish at the boundaries, but this does not help us in the present case.

$$P(x \in [0, L/4]) = \int_0^{L/4} \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^{n\pi/4} \frac{2}{L}\left(\frac{L}{n\pi}\right)\left(\frac{1}{2}\right)(1-\cos 2u)\, du \qquad \left(\text{let } u = \frac{n\pi x}{L}\right)$$

$$= \frac{1}{n\pi}\left[u - \frac{1}{2}\sin 2u\right]_0^{L/4} = \frac{1}{n\pi}\left[\frac{n\pi}{4} - \frac{1}{2}\right] = \frac{1}{4} - \frac{1}{2n\pi} \tag{6}$$

For the ground state, $n=1$, and $P=\frac{1}{4}-\frac{1}{2\pi}\approx 0.091$.

2. The wave function for the ground state of hydrogen ($n=1$) is

$$\psi_1 = \frac{1}{\sqrt{\pi a_o^3}} e^{-r/a_o} \tag{7}$$

where a_o is the Bohr radius.

(a) What is the *most probable* value of r for the ground state?
(b) What is the total probability of finding the electron at a distance greater than this radius?

Solution: The most probable value of r is that which maximizes the probability density,

$$P(r) = 4\pi r^2 |\psi|^2 = \frac{4r^2}{a_o^3} e^{-2r/a_o} \tag{8}$$

Maximizing requires setting $dP/dR == 0$:

$$\frac{dP}{dr} = \frac{8r}{a_o^3} e^{-2r/a_o} - \frac{8r^2}{a_o^4} e^{-2r/a_o} = \frac{8}{a_o^3} e^{-2r/a_o}\left(r - \frac{r_o^2}{a_o}\right) = 0 \tag{9}$$

$$r = \{0, a_o, \infty\} \tag{10}$$

One can readily verify from a quick plot that $r = a_o$ is the maximum, the other two (trivial) extreme values are minima.

The probability of finding the electron at a radius greater than $r = a_o$ is found be integrating the probability density from a_o to ∞:

$$P(x > a_o) = \int_{a_o}^{\infty} P(r)\, dr = \int_{a_o}^{\infty} \frac{4r^2}{a_o^3} e^{-2r/a_o}\, dr = \frac{4}{a-o^3}\left[-\frac{a_o r^2}{2} - \frac{2ra_o^2}{4} - \frac{2a_o^3}{8}\right] e^{-2r/a_o}\Big|_{a_o}^{\infty}$$

$$= \frac{5}{e^2} \approx 0.677 \tag{11}$$

3. Schrödinger's equation for a simple harmonic oscillator reads

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \tag{12}$$

The ground state wave function has the form

$$\psi_o=ae^{-\alpha^2x^2} \tag{13}$$

Determine the value of the constant α and the energy of the state.

Solution: We really just need to substitute into Schrödinger's equation. First, we will need $\frac{\partial^2\psi}{\partial x^2}$:

$$\frac{\partial\psi}{\partial x}=-2\alpha^2xae^{-\alpha^2x} \tag{14}$$

$$\frac{\partial^2\psi}{\partial x^2}=-2\alpha^2ae^{-\alpha^2x}+4\alpha^4x^2ae^{-\alpha^2x}=\psi_o\left(4\alpha^4x^2-2\alpha^2\right) \tag{15}$$

Next, we substitute in to Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\left(4\alpha^4x^2-2\alpha^2\right)\psi_o+\frac{1}{2}m\omega^2x^2\psi_o=E\psi_o \tag{16}$$

If this equation is to have a general solution, the coefficients of the x^2 terms on either side must be the same, and the constant terms on either side must be equal in sum. The quadratic terms give:

$$\frac{4\hbar^2\alpha^4}{2m}=\frac{1}{2}m\omega^2 \qquad\Longrightarrow\qquad \alpha=\sqrt{\frac{m\omega}{2\hbar}} \tag{17}$$

Equating the constant terms:

$$E=\frac{\alpha^2\hbar^2}{m}=\frac{\hbar^2}{m}\left(\frac{m\omega}{2\hbar}\right)=\frac{1}{2}\hbar\omega \tag{18}$$

4. By considering the visible spectrum of hydrogen and He^+, show how you could determine spectroscopically if a sample of hydrogen was contaminated with helium. (Hint: look for differences in the visible emission lines, $\lambda\approx 390\sim 750$ nm. A difference of 10 nm is easily measured.)

Solution: *Pfeffer & Nir 3.4* We know the energies in a hydrogen atom are just $E_n=-13.6\,\text{eV}/n^2$ for a given level n. For the He^+ ion, the only real difference is the extra positive charge in the nucleus. If we have Z positive charges in the nucleus, the energies become $E_n = -13.6\,\text{eV}Z^2/n^2$. For $Z = 2$, we just end up multiplying all the energies by a factor 4. The questions are: does this lead to any new radiative transitions, are they in the visible range, and are they well-separated enough? We can just list the energy levels for the two systems and see what we come up with.

We already know that the visible transitions in Hydrogen occur when excited states relax to the $n = 2$ level, and that for large n the transitions will probably have an energy too high to be in the visible range. Thus, we can probably find a new transition for He^+ by just considering the first several levels alone.

n	H E_n (eV)	He^+ E_n (eV)
1	-13.6	$-13.6 \cdot 4$
2	$-13.6 \cdot \frac{1}{4}$	-13.6
3	$-13.6 \cdot \frac{1}{9}$	$-13.6 \cdot \frac{4}{9}$
4	$-13.6 \cdot \frac{1}{16}$	$-13.6 \cdot \frac{1}{4}$
5	$-13.6 \cdot \frac{1}{25}$	$-13.6 \cdot \frac{4}{25}$

We see a couple of things already. The $n=2$ state for He^+ happens to accidentally have the same energy as the $n=1$ state for H, likewise for the $n=4$ state for He^+ and the $n=2$ state for H. That means that we can't just pick transitions at random, some of them will accidentally have the same energy.

However, the $n = 3$ state for He^+ has the curious fraction $4/9$ in it, which can't possibly occur for H. Transitions into the $n = 3$ state should yield unique energies. Let's compute the visible transitions in hydrogen H, since there are only a few, and see if some He^+ transitions stick out in the in-between wavelengths:

H transition	λ_H (nm)	He^+ transition	λ_{He^+} (nm)
$3 \rightarrow 2$	656	$4 \rightarrow 3$	469
$4 \rightarrow 2$	486	$3 \rightarrow 2$	164
$5 \rightarrow 2$	434		
$6 \rightarrow 2$	410		

Already with just the $4 \rightarrow 3$ transition in He^+, we have an expected emission (or absorption) at 469 nm, a full 17 nm from the nearest H line, and well in the visible range to boot (a nice pretty blue). Should be easy to pick out!

5. A meter stick makes an angle of $30°$ with respect to the x'-axis of O'. What must be the value of v if the meter stick makes an angle of $45°$ with respect to the x-axis of O?

Solution: *Gautreau & Savin 4.3* In the primed frame, the meter stick makes an angle of $30°$ with respect to the x axis, so the projections along the x and y axes are:

$$L'_y = L\sin 30 = \frac{1}{2}L \quad (19)$$

$$L'_x = L\cos 30 = \frac{\sqrt{3}}{2}L \quad (20)$$

In the unprimed frame, the projection along the y axis remains unchanged, since there is no relative motion along that axis. Along the x axis, the projection is shortened by a factor γ due to the relative motion along that axis at velocity v:

$$L_y = L'_y \quad (21)$$

$$L_x = L'_x/\gamma \quad (22)$$

Since the angle the meter stick makes with respect to the x axis in the unprimed frame is 45°,

$$\tan 45 = 1 = \frac{L_y}{L_x} = \frac{\frac{1}{2}L\gamma}{L\frac{\sqrt{3}}{2}} = \frac{\gamma}{\sqrt{3}} \quad (23)$$

Thus, $\gamma = \sqrt{3}$, or $\frac{v}{c} = \sqrt{\frac{2}{3}} \approx 0.816$.

6. A Σ^0 particle at rest decays to a Λ^0 particle and a photon. Determine the energy of the released photon, given that the Σ^0 has rest energy $m_\Sigma c^2 = 1192\,\text{MeV}$ and the Λ^0 has rest energy $m_{\Lambda^0}c^2 = 1116\,\text{MeV}$.

Solution: *Gautreau & Savin 32.7* Conservation of energy and momentum are all that we require. Conservation of energy gives:

$$m_\Sigma c^2 = E_\Lambda + E_\gamma \quad (24)$$

Conservation of momentum is simple, since the Σ^0 is at rest:

$$p_\gamma = -p_\Lambda \quad (25)$$

$$\frac{E_\gamma}{c} = -\gamma m_\Lambda v_\Lambda \quad (26)$$

We can write the Λ^0 total energy as

$$E_\Lambda = \sqrt{p_\Lambda^2 c^2 + m_\Lambda^2 c^4} \quad (27)$$

Substituting $p_\Lambda = E_\gamma/c$,

$$m_\Sigma c^2 = \sqrt{p_\Lambda^2 c^2 + m_\Lambda^2 c^4} + E_\gamma \tag{28}$$

$$m_\Sigma c^2 = \sqrt{E_{gamma}^2 + m_\Lambda^2 c^4} + E_\gamma \tag{29}$$

$$m_\Sigma^2 c^4 - 2m_\Sigma c^2 E_\gamma + E_\gamma^2 = E_\gamma^2 + m_\Lambda^2 c^4 \tag{30}$$

$$2m_\Sigma c^2 E_{gamma} = m_\Sigma^2 c^4 - m_\Lambda^2 c^4 \tag{31}$$

$$E_\gamma = \frac{\left(m_\Sigma c^2\right)^2 - \left(m_\Lambda c^2\right)}{2m_\Sigma c^2} \approx 73.6\,\text{MeV} \tag{32}$$

7. A plane, 300 MHz electromagnetic wave is incident normally on a surface of area $50\,\text{cm}^2$. If the intensity of the wave is $9 \times 10^{-5}\,\text{W/m}^2$,
(a) Determine the rate at which photons strike the surface.
(b) Determine the force on the surface if it is perfectly reflecting.

Solution: *Gautreau & Savin 10.20* The quoted intensity is energy per unit time per unit surface area, and the product of intensity and surface area gives energy per unit time, or power:

$$\mathscr{P} = IA = \left(9 \times 10^5\,[\text{W/m}^2]\right)\left(5 \times 10^{-3}\,[\text{m}^2]\right) = 4.5 \times 10^{-7}\,[\text{W}] = \frac{\Delta E}{\Delta t} \tag{33}$$

The energy per unit time delivered by monochromatic photons[ii] is just the number of photons per unit time multiplied by the energy per photon:

$$\mathscr{P} = hf\frac{\Delta N}{\Delta t} = 1.988 \times 10^{-25}\,[\text{J}]\frac{\Delta N}{\Delta t} \tag{34}$$

Equating our two expressions for power and solving for the number of photons per unit time,

$$\frac{\Delta N}{\Delta t} = \frac{IA}{hf} = \frac{4.5 \times 10^{-7}\,[\text{J/s}]}{1.988 \times 10^{-25}\,[\text{J}]} = 2.26 \times 10^{18}\,[\text{s}^{-1}] \tag{35}$$

8. A 0.3 MeV X-ray photon makes a "head on" collision with an electron initially at rest. Using conservation of energy and momentum, find the recoil velocity of the electron. Check your result with the Compton formula.

Solution: *Gautreau & Savin 12.1* Let E and E' be the initial and final energies of the photon, respectively.

[ii] I.e., photons all of the same frequency/wavelength.

Conservation of energy then gives:

$$E = E' + (\gamma - 1)\, mc^2 \tag{36}$$

For a head-on collision, the photon will recoil in the opposite direction, and the electron along the photon's original direction. Conservation of momentum then yields

$$\frac{E}{c} = -\frac{E'}{c} + \gamma m v \tag{37}$$

Given that E, m, and c are known quantities, simultaneous solution of the two equations above to eliminate E' gives $v = 0.65c$.

9. **(a)** How many different photons can be emitted by hydrogen atoms that undergo transitions from the ground state from the $n=4$ state? **(b)** Enumerate their energies, in electron volts.

Solution: One can brute-force this quickly enough to find that there are 6 transitions. One may also solve the problem for an arbitrary n. More generally, the number of possible transitions is just equal to the number of ways one can choose 2 numbers from a set of n without worrying about their order (i.e., the number of combinations choosing 2 elements from a set of n):

$$(\text{number of different photons}) = \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\,(n-1)}{2} \tag{38}$$

This works because the order does not matter: if we have $n=4$ and pick the pair $(3, 2)$ or $(2, 3)$ we need only count the first ordering, not the second. Hence, we use a combination rather than a permutation. Further, you can easily convince yourself that this includes all possible intermediate states, accounting for multi-step transitions such as $4 \to 3 \to 1$. Given $n=4$, we readily find 10 different transitions from the formula above.

Enumerating, we have the following transitions and photon energies, using $E_n = -13.6\,\text{eV}/n^2$:

$4 \to 3$	0.661 eV
$4 \to 2$	2.55 eV
$4 \to 1$	12.76 eV
$3 \to 2$	1.89 eV
$3 \to 1$	12.09 eV
$2 \to 1$	10.2 eV

10. Neglecting spin, in a strong external magnetic field of 5 T, determine the lines resulting from the $2p \rightarrow 1s$ transition ($\lambda_o = 121.0\,\text{nm}$) in hydrogen. Provide a sketch of the energy levels and their m_l values.

Solution: The 2p level has $l = \{-1, 0, 1\}$, and the levels of different l will experience a Zeeman splitting and shift their energies by $l\mu_B B$. Thus, what is a single energy level in zero magnetic field becomes three distinct levels in a non-zero magnetic field, with energies

$$E_o, E_o \pm \mu_B B \tag{39}$$

where $E_o = hc/\lambda_o$. The 1s level has only $l = 0$, and thus experiences no Zeeman splitting. The new transitions thus have energies $E_o, E_o \pm \mu_B B$ rather than just E_o, so the new wavelengths are

$$\lambda = \left\{\lambda_o, \frac{hc}{E_o - \mu_B B}, \frac{hc}{E_o + \mu_B B}\right\} \approx \{121.0, 121.003, 120.997\}\,\text{nm} \tag{40}$$

11. (a) For a free relativistic quantum particle moving with speed v, the total energy is $E = hf = \hbar\omega = \sqrt{p^2c^2 + m^2c^4}$ and the momentum is $p = h/\lambda = \hbar k = \gamma m v$. For the quantum wave representing the particle, the group speed is $v_g = d\omega/dk$. Prove that the group speed of the wave is the same as the speed of the particle.

(b) It is convenient to describe the motion of an electron (or a hole) in a band by giving it an *effective mass* m^*, defined by

$$\frac{1}{m^*} \equiv \frac{1}{\hbar^2}\frac{d^2E}{dk^2} \tag{41}$$

where $k = 2\pi/\lambda$ is the wave number. For a free electron ($p = \hbar k$), show that $m^* = m$.

Solution: **(a)** We can just brute-force this one. Using the energy equation, we can write ω in terms of k:

$$\omega = \frac{1}{\hbar}\sqrt{\hbar^2k^2c^2 + m^2c^4} \tag{42}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{\hbar}\frac{\frac{1}{2}\left(2\hbar^2kc^2\right)}{\sqrt{\hbar^2k^2c^2 + m^2c^4}} \tag{43}$$

$$= \frac{\hbar kc^2}{\sqrt{\hbar^2k^2c^2 + m^2c^4}} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \sqrt{\frac{p^2c^4}{p^2c^2 + m^2c^4}} \tag{44}$$

In the last line, we substituted back in $p = \hbar k$. If we use $p = \gamma m v$, we can reduce this expression to the desired result:

$$\frac{d\omega}{dk} = \sqrt{\frac{p^2c^4}{p^2c^2 + m^2c^4}} = \sqrt{\frac{\gamma^2m^2v^2c^4}{\gamma^2m^2v^2c^2 + m^2c^4}} = \sqrt{\frac{\gamma^2c^2v^2}{\gamma^2v^2 + c^2}} \tag{45}$$

$$= \sqrt{\frac{\frac{c^2v^2}{1-v^2/c^2}}{\frac{v^2}{1-v^2/c^2} + \frac{c^2\left(1-v^2/c^2\right)}{1-v^2/c^2}}} = \sqrt{\frac{c^2v^2}{v^2 + c^2 - v^2}} = \pm v \tag{46}$$

$$\therefore \quad |v_g| = |v| \tag{47}$$

(b) First, d^2E/dk^2:

$$E = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m} \tag{48}$$

$$\frac{dE}{dk} = \frac{\hbar^2k}{m} \tag{49}$$

$$\frac{d^2E}{dk^2} = \frac{\hbar^2}{m} \tag{50}$$

Second, substitution:

$$\frac{1}{m^*} \equiv \frac{1}{\hbar^2}\frac{d^2E}{dk^2} = \frac{1}{\hbar^2}\frac{\hbar^2}{m} = \frac{1}{m} \tag{51}$$

12. An interstellar space probe is moving at a constant speed relative to earth of $0.76c$ toward a distant planet. Its radioisotope generators have enough energy to keep its data transmitter active continuously for 15 years, as measured in their own reference frame.

(a) How long do the generators last as measured from earth?
(b) How far is the probe from earth when the generators fail, as measured from earth?
(c) How far is the probe from earth when the generators fail, *as measured by its built-in trip odometer*?

Solution: Just to be clear, we will label quantities measured in the earth's reference frame with primes ($'$), and quantities without primes are with respect to the probe's reference frame. The relative velocity between the earth and the probe is the same from both reference frames, $v = v'$. From the probe's (and its generators') reference frame, it is the observers on earth that are moving. The observers on earth should then see a *longer* time interval compared to the proper time measured on the probe:

$$\Delta t' = \gamma\Delta_p = \frac{15\,\text{yrs}}{\sqrt{1 - \frac{(0.76c)^2}{c^2}}} \approx 23\,\text{yrs}$$

According to observers on earth, the generators should fail after a period of $\Delta t'$. Also according to them, the probe should have traveled a distance $d' = v'\Delta t'$ - the earth-bound observers watched the probe travel for an interval $\Delta t'$ at a constant velocity of v' in their reference frame:

$$d' = v'\Delta t' = (23\,\text{yrs})\left(0.76 \times 3 \times 10^8\,\text{m/s}\right) \approx 1.65 \times 10^{17}\,\text{m}$$

Alternatively, we could express the distance in light years - the distance light travels in one year:

$$d' = (0.76\,\text{light speed})\,(23\,\text{yrs}) \approx 17.5\,\text{light-years}$$

Finally, how about the distance traveled according to the probe? That is just the relative velocity multiplied by the elapsed time *from the probe's reference frame, i.e.*, the proper time:

$$d = v\Delta t = (15\,\text{yrs})\left(3 \times 10^8\,\text{m/s}\right)(0.76) = 1.1 \times 10^{17}\,\text{m} = 11\,\text{light-years}$$