University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Spring 2010

Problem Set 5

Instructions:

1. Answer all questions below.
2. Show your work for full credit.
3. All problems are due Fri 26 February 2010 by the end of the day.
4. You may collaborate, but everyone must turn in their own work.

Note: you can turn in exam problem 4b for bonus exam credit along with this homework.

1. Solve one of the exam problems that you did not choose.

2. *The Thompson model of the atom.* Show that in the Thompson hydrogen atom, the net force exerted on the electron is directly proportional to its displacement from the center of the atom (i.e., it undergoes simple harmonic motion). Calculate the frequency of radiation that would be emitted by the vibration of the electron. Assume the atom's radius is 10^{-10} m.

3. *Quantum particle in a box.* In classical mechanics, an electron with energy E in a one-dimensional box of length L simply bounces back and forth between the walls at constant speed. A classical electron thus has an equal probability of being anywhere in the box; the probability per unit length for an electron to be at any point is then $P_{\text{classical}} = 1/L$. For example, classical mechanics predicts that an electron would be found in the middle half of the box with a probability of exactly 1/2, or 50% of the time.

(a) What are the normalized wavefunctions for an electron in a one-dimensional box of width L?
(b) In the quantum-mechanical ground state (lowest energy state), what is the probability per unit length that the electron is at the center of the box?
(c) What is the probability of finding the electron in the middle half of the box (i.e., from $L/4$ to $3L/4$)?

4. *Reduced-mass correction to the hydrogen spectrum.* In our calculation of the energies of the stationary states of hydrogen, we pretended that the proton remains at rest. Actually, both the electron and proton orbit about their common center of mass. Show that the energies of the stationary states, taking into account this motion of the proton, are given by

$$E_n = -\frac{me^4}{2\left(4\pi\epsilon_o\right)^2 \hbar^2 n^2} \tag{1}$$

where m is the reduced mass

$$m = \frac{m_e m_p}{m_e + m_p} \tag{2}$$

What does this imply about the spectrum of the deuterium spectrum? (Deuterium is hydrogen with a neutron and a proton in the nucleus.)

Hint: The electron and proton move in circles of radii

$$r_e = r\frac{m_p}{m_p + m_e} \qquad \textit{and} \qquad r_p = r\frac{m_e}{m_p + m_e} \tag{3}$$

where r *is the distance between the electron and proton. According to Bohr's theory, the net angular momentum of this system of two particles is quantized,* $L = n\hbar$.

5. *Why can bound-state wave functions be chosen to be real?* In a one-dimensional problem, the spatial wave functions for any allowed state can be chosen to be real-valued. Verify this using the following outline, or some other method.

(a) Write the wave function $\psi_n(x)$ in terms of its real and imaginary parts: $\psi_n = \mathrm{Re}(\psi_n) + i\,\mathrm{Im}(\psi_n)$, and substitute this into the Schrödinger equation.
(b) Show that $\mathrm{Re}(\psi_n)$ and $\mathrm{Im}(\psi_n)$ separately satisfy the Schrödinger equation.
(c) In one dimension, there is only one (linearly independent) wave function for each energy E_n. What does this imply about $\mathrm{Re}(\psi_n)$ and $\mathrm{Im}(\psi_n)$?

6. *Normalization of a wave function.* A particle bound in a certain one-dimensional potential has a wave function described by the following equations:

$$\psi(x) = 0 \qquad x < -\frac{L}{2} \tag{4}$$

$$\psi(x) = Ae^{ikx}\cos\frac{3\pi x}{L} \qquad -\frac{L}{2} \leqslant x \leqslant \frac{L}{2} \tag{5}$$

$$\psi(x) = 0 \qquad x > \frac{L}{2} \tag{6}$$

(a) Find the value of the normalization constant A by enforcing the condition $\int_{\text{all } x} |\psi(x)|^2\,dx = 1$.
(b) What is the probability that the particle will be found between $x = 0$ and $x = L/4$?

7. *Quantum harmonic oscillator.* The harmonic oscillator potential is $U(x) = \frac{1}{2}m\omega_o^2 x^2$; a particle of mass m in this potential oscillates with frequency ω_o. The ground state wave function for a particle in the harmonic oscillator potential has the form

$$\psi(x) = Ae^{-ax^2} \tag{7}$$

(a) By substituting $U(x)$ and $\psi(x)$ into the one-dimensional time-independent Schrödinger equation, find expressions for the ground-state energy E and the constant a in terms of m, $\hbar$, and ω_o.
(b) Apply the normalization condition to determine the constant A in terms of m, $\hbar$, and ω_o.