

Problem Set 5: Solutions

1. Solve one of the exam problems that you did not choose.
2. *The Thompson model of the atom.* Show that in the Thompson hydrogen atom, the net force exerted on the electron is directly proportional to its displacement from the center of the atom (i.e., it undergoes simple harmonic motion). Calculate the frequency of radiation that would be emitted by the vibration of the electron. Assume the atom's radius is 10^{-10} m.

In the Thompson model, the proton, having total charge $+e$ is imagined to be a sphere of radius r_o uniform charge density. Inside this sphere of positive charge, the electron sits a distance $r < r_o$ from the center of the sphere.

At a distance $r < r_o$, Gauss' law tells us that the force on the electron is only due to the amount of positive charge contained within a radius r .ⁱ If the electron were outside the sphere, it would be the whole charge of the proton e , but inside the sphere, only charge contained in $r < r_o$ contributes. Since the sphere has uniform charge density, the amount of positive charge within r is just the volume fraction of the sphere contained in r times the total charge of the proton:

$$q(r) = -e \left(\frac{4\pi r^3}{4\pi r_o^3} \right) = -e \frac{r^3}{r_o^3} \tag{1}$$

The net force on the electron (which has charge $-e$) is then

$$F(r) = \frac{-k_e e^2 r^3}{r^2 r_o^3} = \frac{-k_e e^2}{r_o^3} r = m a_r \quad \implies \quad a_r = -\frac{k_e e^2}{m r_o^3} r \tag{2}$$

The acceleration along the radial direction is proportional to the radial position, with a negative sign, which means that the electron undergoes simple harmonic motion. The farther the electron gets from the center of the proton, the more of the proton's charge there is pulling it back – the electric force looks just like a spring's restoring force. Any time we can show $a_x = -\omega^2 x$, it means there is simple harmonic motion along the x direction with angular frequency $\omega = 2\pi f$. Thus,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k_e e^2}{m r_o^3}} \approx 2.53 \times 10^{15} \text{ Hz} \quad \implies \quad \lambda = \frac{c}{f} \approx 118 \text{ nm} \tag{3}$$

ⁱThis should be familiar from PH106 ...

This is unreasonable for many reasons, and we can come up with two easy ones. First, we only get a single frequency of oscillation in this model, and hence a single emitted wavelength, whereas we know actual atomic spectra for even simple atoms have several (or many) emitted wavelengths. Second, we could consider the energetics of the atom, since we have shown above it behaves as a simple harmonic oscillator.

If we give the electron one quantum of energy, $\hbar\omega$, the lowest possible amount according to Planck's hypothesis, we can calculate the implied amplitude of the electron's oscillation, since we know that a simple harmonic oscillator's energy can be related to amplitude A and angular frequency: $\frac{1}{2}m\omega^2A^2$. Using our expression for ω above,

$$E = \frac{1}{2}m\omega^2A^2 = \hbar\omega \quad \Longrightarrow \quad A^2 = \sqrt{\frac{2\hbar}{m\omega}} = 2\hbar\sqrt{\frac{r_0^3}{k_e e^2 m}} \quad \Longrightarrow \quad A \approx 1.2r_0 \quad (4)$$

So: even with a single quantum of energy, the electron's amplitude of oscillation would put it outside the proton! Clearly, this is a problem for the Thompson model.

3. Quantum particle in a box. In classical mechanics, an electron with energy E in a one-dimensional box of length L simply bounces back and forth between the walls at constant speed. A classical electron thus has an equal probability of being anywhere in the box; the probability per unit length for an electron to be at any point is then $P_{\text{classical}}=1/L$. For example, classical mechanics predicts that an electron would be found in the middle half of the box with a probability of exactly $1/2$, or 50% of the time.

- (a) What are the normalized wavefunctions for an electron in a one-dimensional box of width L ?
- (b) In the quantum-mechanical ground state (lowest energy state), what is the probability per unit length that the electron is at the center of the box?
- (c) What is the probability of finding the electron in the middle half of the box (i.e., from $L/4$ to $3L/4$)?

Within the box ($|x| \leq L$), our particle obeys the free-particle Schrödinger equation, since the potential is zero. Outside the box ($|x| > L$) the potential is infinite. Since the particle cannot exist there, its wavefunction must vanish in this region. The free-particle Schrödinger equation inside the box reads

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E\psi \quad (5)$$

This is nothing more than the 1-D wave equation we know from simple harmonic motion, if we re-group

the terms:

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2} \psi \quad (6)$$

In which case we know what the general solution must be:

$$\psi(x) = A \cos kx + B \sin kx \quad \text{with} \quad k = \sqrt{\frac{2mE}{\hbar^2}} \quad (7)$$

This solution is subject to the boundary conditions that ψ and its derivative are continuous at the $|x| \leq L$ boundaries, i.e, both ψ and its first derivative vanish at $|x|=L$:

$$\psi(0) = \frac{\partial \psi}{\partial x}(0) = \psi(L) = \frac{\partial \psi}{\partial x}(L) = 0 \quad (8)$$

The first conditions immediately lead us to the conclusion that $B=0$. The second condition implies

$$kL = n\pi \quad \implies \quad k = \frac{n\pi}{L} \quad (9)$$

Our solution is then

$$\psi(x) = A \sin\left(\frac{n\pi x}{L}\right) \quad (10)$$

while Eq. 7 implies

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L} \quad \implies \quad E = \frac{\hbar^2 n^2 \pi^2}{2m} \quad (11)$$

which was our previous result from the Bohr model. We have yet to determine the constant A , however. This we can do by enforcing unit probability of finding the particle *somewhere*, i.e., by integrating $|\psi|^2$ over all x . Since ψ is zero outside of the region $0 \leq x \leq L$, we need only integrate over that range.

$$1 = \int_0^L |\psi|^2 dx = \int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = A^2 \left[\frac{x}{2} - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right) \right]_0^L = A^2 \left[\frac{L}{2} \right] \quad (12)$$

$$\implies \quad A = \sqrt{\frac{2}{L}} \quad (13)$$

Thus, the normalized wavefunction is

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad (14)$$

For the ground state, $n = 1$. The probability per unit length of the electron being found in any interval $[x, x + dx]$ is $P(x) = |\psi(x)|^2$. The probability that the electron is to be found at the center of the box ($x = L/2$) is just this probability evaluated at $x = L/2$:

$$P(x) = |\psi(x)|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) \quad (15)$$

$$P\left(\frac{L}{2}\right) = \frac{2}{L} \sin^2\left(\frac{n\pi L}{2L}\right) = \frac{2}{L} \quad (16)$$

As one might expect, the probability is inversely proportional to the width of the box – the wider the box, the smaller the chance that one finds the electron in the middle. Somewhat surprisingly, the quantum prediction is a factor 2 larger than the classical prediction of $1/L$.

The probability of finding the particle in the middle half of the box, $L/4 \leq x \leq 3L/4$, means integrating the probability per unit length over that interval. Making use of the indefinite integral given above,

$$P_{\text{mid half}} = \int_{L/4}^{3L/4} |\psi(x)|^2 dx = \int_{L/4}^{3L/4} \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} + \frac{1}{\pi} \approx 0.818 \quad (17)$$

Contrast this with the classical prediction of 50%.

4. Reduced-mass correction to the hydrogen spectrum. In our calculation of the energies of the stationary states of hydrogen, we pretended that the proton remains at rest. Actually, both the electron and proton orbit about their common center of mass. Show that the energies of the stationary states, taking into account this motion of the proton, are given by

$$E_n = -\frac{m e^4}{2(4\pi\epsilon_0)^2 \hbar^2 n^2} \quad (18)$$

where m is the reduced mass

$$m = \frac{m_e m_p}{m_e + m_p} \quad (19)$$

What does this imply about the spectrum of the deuterium spectrum? (Deuterium is hydrogen

with a neutron and a proton in the nucleus.)

Hint: The electron and proton move in circles of radii

$$r_e = r \frac{m_p}{m_p + m_e} \quad \text{and} \quad r_p = r \frac{m_e}{m_p + m_e} \quad (20)$$

where r is the distance between the electron and proton. According to Bohr's theory, the net angular momentum of this system of two particles is quantized, $L = n\hbar$.

There is a fairly good chance that you've not encountered the reduced mass in your previous mechanics classes, so we will go through it quickly. Essentially, the reduced mass is an "effective" inertial mass that can be used solve a two-body problem as if it is a single body problem.

Our system consists of a single proton and a single electron, separated by a distance r described by a vector \vec{r} . Classically, we expect both particles to orbit about their center of mass. The proton, being much heavier, will move very little in comparison to the lighter electron, and sit closer to the center of mass. This is depicted below.ⁱⁱ

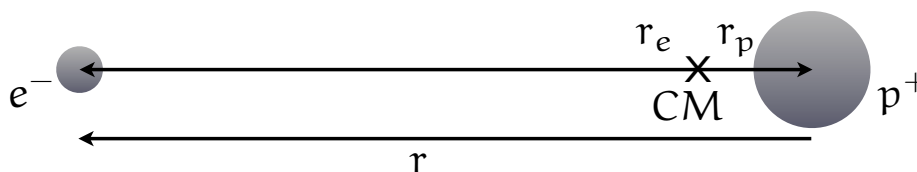


Figure 1: An electron and a proton, with a separation \vec{r} . The electron sits at a distance r_e from their center of mass, and the proton a distance r_p from the center of mass.

The center of mass for a simple two-body system like this essentially amounts to finding the point

ⁱⁱSome neat animations can be found at <http://en.wikipedia.org/wiki/Barycenter>.

along their common axis at which their torques would balance.ⁱⁱⁱ If the particles' positions are described by vectors \vec{r}_e and \vec{r}_p pointing from the center of mass toward the electron and proton, respectively, then this requires

$$m_e \vec{r}_e + m_p \vec{r}_p = 0 \quad \text{or} \quad \vec{r}_p = -\frac{m_e}{m_p} \vec{r}_e \quad (21)$$

Of course, we also know that the $\vec{r} = \vec{r}_e - \vec{r}_p$. Rearranging the first equation and substituting,

$$\vec{r} = \vec{r}_e - \vec{r}_p = \vec{r}_e \left(1 + \frac{m_e}{m_p} \right) = -\vec{r}_p \left(\frac{m_e}{m_p} + 1 \right) \quad (22)$$

This leads us to expressions for the electron and proton orbital radii in terms of their separation and masses:

$$\vec{r}_e = \vec{r} \left(\frac{m_p}{m_p + m_e} \right) \quad \vec{r}_p = -\vec{r} \left(\frac{m_e}{m_p + m_e} \right) \quad (23)$$

Here the negative sign for \vec{r}_p just indicates that the direction is opposite \vec{r} . Two other interesting things to note: first, only the relative mass m_e/m_p really matters; second, owing to the fact that $m_p \gg m_e$, the proton will orbit in a much smaller circle than the electron. So much smaller, in fact, that we are usually well-justified in taking the proton to be at rest. Now, let us write down the kinetic energy of the two-particle system. We'll have the kinetic energy of each particle, and their electrical potential energy :

$$E = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p v_p^2 - \frac{k_e e^2}{r} = \frac{1}{2} m_e \left(\frac{dr_e}{dt} \right)^2 + \frac{1}{2} m_p \left(\frac{dr_p}{dt} \right)^2 - \frac{k_e e^2}{r} \quad (24)$$

Using the expressions above, we can write the individual particles' velocities in terms of the rate at which the separation changes with time:

$$v_e = \frac{dr_e}{dt} = \left(\frac{dr}{dt} \right) \left(\frac{m_p}{m_p + m_e} \right) \quad v_p = \frac{dr_p}{dt} = \left(\frac{dr}{dt} \right) \left(\frac{m_e}{m_p + m_e} \right) \quad (25)$$

But $d\vec{r}/dt$ is nothing more than the relative velocity of the proton and electron, which we'll simply call \vec{v} :

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (\vec{r}_e - \vec{r}_p) = \frac{d\vec{r}_e}{dt} - \frac{d\vec{r}_p}{dt} \quad (26)$$

ⁱⁱⁱPut more simply, if the two particles were on a see-saw, where should the center pivot be?

With this definition, we may write the electron and proton velocities as^{iv}

$$v_e = v \left(\frac{m_p}{m_p + m_e} \right) \quad v_p = v \left(\frac{m_e}{m_p + m_e} \right) \quad (27)$$

Substituting these expressions into our energy equation,

$$E = \frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p v_p^2 - \frac{k_e e^2}{r} = \frac{1}{2} m_e v^2 \frac{m_p^2}{(m_p + m_e)^2} + \frac{1}{2} m_p v^2 \frac{m_e^2}{(m_p + m_e)^2} - \frac{k_e e^2}{r} \quad (28)$$

$$E = \frac{m_e m_p}{2(m_e + m_p)^2} (m_p v^2 + m_e v^2) - \frac{k_e e^2}{r} = \frac{1}{2} \left(\frac{m_e m_p}{m_e + m_p} \right) v^2 - \frac{k_e e^2}{r} \quad (29)$$

If we define a quantity $m = m_e m_p / (m_e + m_p)$, known as the “reduced mass,” our energy equation is identical to that of a single particle of charge $-e$ in an electric potential $V = k_e e / r$:

$$E = \frac{1}{2} m v^2 - eV \quad (30)$$

We have successfully reduced a two-body problem to a single-body problem. How about the angular momentum? Taking the center of mass as our origin, the net angular momentum is mvr for each particle:

$$L = L_p + L_e = m_e \left(\frac{dr_e}{dt} \right) r_e + m_p \left(\frac{dr_p}{dt} \right) r_p \quad (31)$$

$$L = m_e \left(\frac{vm_p}{m_p + m_e} \right) \left(\frac{rm_p}{m_p + m_e} \right) + m_p \left(\frac{-m_e v}{m_e + m_p} \right) \left(\frac{-rm_e}{m_p + m_e} \right) \quad (32)$$

$$L = \frac{m_e m_p v r}{(m_e + m_p)^2} (m_e + m_p) = \left(\frac{m_e m_p}{m_e + m_p} \right) v r = mvr \quad (33)$$

Again, we have reduced a two-body problem to an equivalent single-body problem: the total angular momentum is simply the reduced mass times the relative velocity times the separation distance. We can now apply our quantization condition for the total angular momentum:

$$mvr = n\hbar \quad \text{or} \quad v = \frac{n\hbar}{mr} \quad (n \in \mathbb{N}_1 = \{1, 2, \dots\}) \quad (34)$$

Substituting this condition into our energy equation:

$$E = \frac{1}{2} m \left(\frac{n^2 \hbar^2}{m^2 r^2} \right) - \frac{k_e e^2}{r} = \frac{n^2 \hbar^2}{2mr^2} - \frac{k_e e^2}{r} \quad (35)$$

^{iv}We don't really need the vector notation any more, since we're just going to square them anyway.

The definition of a stationary state is a state in which the energy has no time dependence. In this case, there is no explicit time dependence, but an equivalent condition is to require that the variation with respect to radius vanishes:

$$\frac{dE}{dt} = \frac{dE}{dr} \frac{dr}{dt} = 0 = \left(\frac{-n^2 \hbar^2}{mr^3} + \frac{ke^2}{r^2} \right) \frac{dr}{dt} \quad (36)$$

$$\implies \frac{n^2 \hbar^2}{mr} = ke^2 \quad \text{or} \quad r = \frac{n^2 \hbar^2}{kme^2} \quad (37)$$

Using this extremal radius in our energy equation, we find

$$E = \frac{n^2 \hbar^2}{2m} \left(\frac{k^2 m^2 e^4}{n^4 \hbar^4} \right) - \frac{k^2 m e^4}{n^2 \hbar^2} \quad (38)$$

$$E = \frac{-k^2 m e^4}{2n^2 \hbar^2} = \frac{-me^4}{2(4\pi\epsilon_0)^2 n^2 \hbar^2} \quad (39)$$

For the last line, we noted that $k_e = 1/4\pi\epsilon_0$. This implies that the hydrogen and deuterium^v nuclei will have slightly different emission and absorption lines, owing to the fact that the deuterium nucleus is about twice as heavy as the hydrogen nucleus. The deuterium transitions should be at slightly higher energies (lower wavelengths). It is a tiny effect, about a 0.2 nm shift of the 656.3 nm H line, but quite measurable and in good agreement with this simple model.^{vi} Similar effects are expected and observed for other hydrogen-like systems, e.g., He^+ and Li^{++} .

5. *Why can bound-state wave functions be chosen to be real?* In a one-dimensional problem, the spatial wave functions for any allowed state can be chosen to be real-valued. Verify this using the following outline, or some other method.

(a) Write the wave function $\psi_n(x)$ in terms of its real and imaginary parts: $\psi_n = \text{Re}(\psi_n) + i \text{Im}(\psi_n)$, and substitute this into the Schrödinger equation.

(b) Show that $\text{Re}(\psi_n)$ and $\text{Im}(\psi_n)$ separately satisfy the Schrödinger equation.

(c) In one dimension, there is only one (linearly independent) wave function for each energy E_n . What does this imply about $\text{Re}(\psi_n)$ and $\text{Im}(\psi_n)$?

We should remember that for any complex function, both the independent and dependent variables can be separated into real and imaginary parts. That is, generally speaking, we can decompose a complex function w into a two real-valued functions u and v :

$$z = x + iy \quad (40)$$

$$w = f(z) = u(x, y) + iv(x, y) \quad x, y \in \mathbb{R} \quad (41)$$

^vDeuterium is hydrogen with a neutron in the nucleus in addition to the proton.

^{vi}See, for example, <http://hyperphysics.phy-astr.gsu.edu/Hbase/quantum/hydfin.html>.

That means that our complex-valued wavefunction ψ can always be decomposed into real and imaginary parts ψ_r and ψ_i , respectively:

$$\psi = \psi_r + i\psi_i \quad (42)$$

Plugging this into the one-dimensional Schrödinger equation,

$$E\psi = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U(x)\psi \quad (43)$$

$$E(\psi_r + i\psi_i) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} (\psi_r + i\psi_i) + U(x) (\psi_r + i\psi_i) \quad (44)$$

$$E\psi_r + iE\psi_i = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi_r}{\partial x^2} + i \left(\frac{-\hbar^2}{2m} \frac{\partial^2 \psi_i}{\partial x^2} \right) + U(x)\psi_r + iU(x)\psi_i \quad (45)$$

We can now separate out the real and imaginary parts of both sides and make two separate equations (since anything with an i in front of it can never equal something without an i at all):

$$E\psi_r = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi_r}{\partial x^2} + U(x)\psi_r \quad (46)$$

$$E\psi_i = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi_i}{\partial x^2} + U(x)\psi_i \quad (47)$$

Lo and behold, both the real and imaginary parts of the wavefunction must separately satisfy the same Schrödinger equation. If there is to be only a single linearly-independent solution, then the real and imaginary parts must clearly be equal, $\psi_r = \psi_i$.

6. Normalization of a wave function. A particle bound in a certain one-dimensional potential has a wave function described by the following equations:

$$\psi(x) = 0 \quad x < -\frac{L}{2} \quad (48)$$

$$\psi(x) = Ae^{ikx} \cos \frac{3\pi x}{L} \quad -\frac{L}{2} \leq x \leq \frac{L}{2} \quad (49)$$

$$\psi(x) = 0 \quad x > \frac{L}{2} \quad (50)$$

- (a) Find the value of the normalization constant A by enforcing the condition $\int_{\text{all } x} |\psi(x)|^2 dx = 1$.
 (b) What is the probability that the particle will be found between $x=0$ and $x=L/4$?

This should seem familiar: we just did it a few problems ago. Thus we know that $A = \sqrt{2/L}$ already. Still ... practice makes perfect:^{vii}

^{vii}And, in the previous incarnation, you did not need to remember that $|e^{ix}|^2 = 1$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{-L/2} 0 dx + \int_{-L/2}^{L/2} \left| A e^{ikx} \cos\left(\frac{3\pi x}{L}\right) \right|^2 dx + \int_{L/2}^{\infty} 0 dx \\
1 &= A^2 \int_{-L/2}^{L/2} \cos^2\left(\frac{3\pi x}{L}\right) dx \quad \text{let } u = \frac{3\pi x}{L} \implies dx = \frac{L}{3\pi} du \\
1 &= A^2 \left(\frac{L}{3\pi}\right) \int_{-3\pi/2}^{3\pi/2} \cos^2 u du = \frac{A^2 L}{6\pi} \left[u + \sin u \cos u \right]_{-3\pi/2}^{3\pi/2} = \frac{A^2 L}{3\pi} \left(\frac{3\pi}{2}\right) = \frac{1}{2} A^2 L \\
\implies A &= \sqrt{\frac{2}{L}} \tag{51}
\end{aligned}$$

Same box, same normalization constant. The probability that the particle is found between 0 and $L/4$ is obtained by integrating the square of the wave function between those limits, rather than over all infinity:

$$\begin{aligned}
P &= \int_0^{L/4} |\psi(x)|^2 dx = \int_0^{L/4} \left| \sqrt{\frac{2}{L}} e^{ikx} \cos\left(\frac{3\pi x}{L}\right) \right|^2 dx = \frac{2}{L} \int_0^{L/4} \cos^2 \frac{3\pi x}{L} dx \\
P &= \left(\frac{2}{L}\right) \left(\frac{L}{3\pi}\right) \int_0^{3\pi/4} \cos^2 u du = \frac{1}{3\pi} \left[u + \sin u \cos u \right]_0^{3\pi/4} = \frac{1}{4} - \frac{1}{6\pi} \approx 0.2 \tag{52}
\end{aligned}$$

7. Quantum harmonic oscillator. The harmonic oscillator potential is $U(x) = \frac{1}{2} m \omega_0^2 x^2$; a particle of mass m in this potential oscillates with frequency ω_0 . The ground state wave function for a particle in the harmonic oscillator potential has the form

$$\psi(x) = A e^{-\alpha x^2} \tag{53}$$

- (a) By substituting $U(x)$ and $\psi(x)$ into the one-dimensional time-independent Schrödinger equation, find expressions for the ground-state energy E and the constant α in terms of m , \hbar , and ω_0 .
(b) Apply the normalization condition to determine the constant A in terms of m , \hbar , and ω_0 .

Given the form of ψ , we note

$$\frac{\partial^2 \psi}{\partial x^2} = (4\alpha^2 x^2 - 2\alpha) \psi \tag{54}$$

And thus

$$E\psi = \frac{-\hbar^2}{2m} (4a^2x^2 - 2a) \psi + \frac{1}{2}m\omega_o^2x^2\psi \quad (55)$$

$$\text{or } E\psi = \left(\frac{1}{2}m\omega_o^2 - \frac{2a^2\hbar^2}{m} \right) x^2\psi + \frac{a\hbar^2}{m}\psi \quad (56)$$

If we are to have a unique solution for all x , the x^2 coefficients must vanish, i.e.,

$$\frac{1}{2}m\omega_o^2 - \frac{2a^2\hbar^2}{m} = 0 \quad \implies \quad a = \frac{m\omega_o}{2\hbar} \quad (57)$$

Similarly, the terms of the form (constant) ψ must equate, giving:

$$E = \frac{a\hbar^2}{m} = \frac{1}{2}\hbar\omega_o \quad (58)$$

This is precisely what we have found before, the ground state energy of the harmonic oscillator is half the Planck energy quantum. Normalization requires that

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \quad (59)$$

Noting that

$$\int_{-\infty}^{\infty} e^{-cx^2} dx = \sqrt{\frac{\pi}{c}} \quad (60)$$

we have

$$1 = \int_{-\infty}^{\infty} A^2 e^{-2ax^2} dx = A^2 \sqrt{\frac{\pi}{2a}} \quad (61)$$

$$\implies A^2 = \sqrt{\frac{2a}{\pi}} \quad \text{or} \quad A = \sqrt[4]{\frac{2a}{\pi}} = \sqrt[4]{\frac{m\omega_o}{\pi\hbar}} \quad (62)$$

The ground state wave function is thus

$$\psi(x) = \sqrt[4]{\frac{m\omega_o}{\pi\hbar}} e^{-\left(\frac{m\omega_o}{2\hbar}\right)x^2} \quad (63)$$