University of Alabama
Department of Physics and Astronomy

PH 105 / LeClair Summer 2012

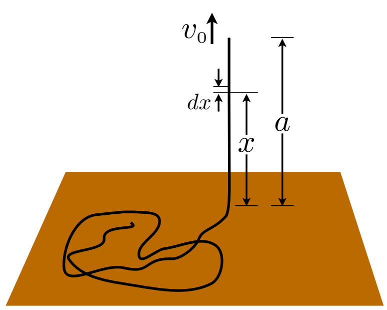
Exam II practice

1. A uniformly dense rope of length b and mass per unit length λ is coiled on a smooth table. One end is lifted by hand with constant velocity v_o. Find the force of the rope held by the hand when the rope is a distance a above the table ($b > a$).

Find: The force a rope exerts on a hand pulling it upward off of a table, as a function of position. The hand will have to pull against the rope's weight, but also against the changing momentum of the rope as more of it leaves the table.

Given: The length b and linear mass density λ, the constant velocity at which the rope is pulled.

Sketch: We want to know the total force between the hand and rope when a length a of the rope has been pulled off of the table at constant speed v_o.



Take a small segment of rope dx a height x off of the table, as shown in the sketch above, with the $+x$ direction being upward. This small segment has mass $dm = \lambda dx$, and was pulled off of the table at constant velocity v_o. Just before the segment was pulled off of the table, it was simply lying there with zero velocity and hence zero momentum. An instant later, it is moving away from the table at velocity v_o, which clearly implies a non-zero momentum. This means that during the time dt it took to pull the segment dx off of the table completely, its momentum changed from 0 to p_f. This time rate of change of momentum is a force.

Relevant equations: The main equation we will need is that force is the time rate of change of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Additionally, we need to know the weight of an arbitrary length of rope. Take a small section of rope of length a. Since the mass per unit length of the rope is λ, the mass of that segment must be λa, and its weight $\lambda g dx$.

Symbolic solution: Consider again our segment of rope dx. It has mass dm and velocity v_o just after it leaves the table, and zero velocity just before. The momentum change dp in pulling that segment of rope off of the table is

$$dp = v_o dm = v_o \lambda dx$$

If this segment took dp to pull off of the table, we can easily find the time rate of change of momentum by dividing by dt:

$$\frac{dp}{dt} = v_o \lambda \frac{dx}{dt} = v_o^2 \lambda$$

Here we used the fact that dx/dt is simply the velocity of the rope, which were are given as v_o. This is the impulse force that brings the string off the table, and which also acts on the hand pulling it off of the table. This impulse force is independent of how much rope is already off of the table, which makes sense: it only involves changing the momentum of an infinitesimal bit of rope at one instant, and does not depend on what the rest of the rope is doing. Since the bit of rope changes its velocity from zero to straight upward, the impulse that the hand feels must act in the downward direction by Newton's third law. That is, the force acting on the hand F_i must be equal and opposite of the impulse force acting on the rope, which is equal to the rope's time rate of change in momentum:

$$F_i = -\frac{dp}{dt} = -v_o^2 \lambda$$

In addition to the impulse, the hand must also support the weight of the string already off of the table. A length a of the rope must have mass λa, and therefore the hand must support a weight of $W = -\lambda g a$, also acting downward. The total force on the hand is this weight plus the impulse force:

$$F_{\text{tot}} = W + F_i = -\lambda g a - \lambda v_o^2 = -\lambda g a \left(1 + \frac{v_o^2}{a g}\right)$$

Numeric solution: Once again, there are no numbers given.
Double check: Dimensionally, our answer is correct. Checking each term in our force balance, noting that λ must have units of kilograms per meter

$$\begin{aligned}\lambda g a &= \left[\mathrm{kg\,m^{-1}}\right]\left[\mathrm{m\,s^2}\right][\mathrm{m}] = \left[\mathrm{kg\,m/s^2}\right] = [\mathrm{N}] \\ \lambda v_o^2 &= \left[\mathrm{kg\,m^{-1}}\right]\left[\mathrm{m^2\,s^2}\right] = [\mathrm{N}]\end{aligned}$$

Our answer also makes sense qualitatively: both the impulse and weight force should get larger as λ increases (i.e., as the rope gets heavier). As the total length of rope above the table a increases, the weight should increase while the impulse force remains constant, which also makes sense. Finally, the impulse force should increase as the pulling speed v_o increases, while the weight should be unaffected.

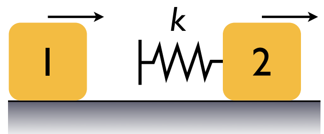
2. Block 1 of mass m_1 is moving rightward at v_1 while block 2 of mass m_2 is moving rightward at $v_2 < v_1$. The surface is frictionless, and a spring of constant k is fixed to block 2. When the blocks collide, the compression of the spring is maximum the instant the blocks have the same velocity.

(a) Show that

$$\Delta K = K_{1i} + K_{2i} - K_{12} = \frac{1}{2}\mu v_{\text{rel}}^2 \qquad \text{with} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

where K_{1i} and K_{2i} are the kinetic energies of blocks 1 and 2 before the collision, respectively, K_{12} is the kinetic energy of the system at the moment the spring compression is maximum, and v_{rel} is the relative velocity of the two blocks. The quantity μ is known as the *reduced mass* of the system.

(b) Find the maximum compression of the spring.



Just for fun, we will solve this one two ways: from the usual "laboratory frame" where we watch both blocks from the floor, and a frame of reference where block 2 is stationary. The latter is quite a bit less messy . . . but does require the foresight to think of it in the first place.

Find: The chance in kinetic energy of the blocks between the moment just before their collision and at the moment the spring is at maximum compression, which is also the point at which the two blocks have equal speeds. We must also find the maximum compression of the spring.

Given: A collision between two blocks, one of which has a spring connected to it. We know the block's initial speeds and the spring constant.

Sketch: We really don't need anything beyond what is given.

Relevant equations: Owing to the spring force present, we cannot apply conservation of kinetic energy, meaning we cannot use our equations for elastic collisions. However, since there is no friction, and we are not asked to consider what happens after the spring reaches maximum compression,[i] we can use conservation of total energy, including the spring's potential energy U_s:

$$K_{1i} + K_{2i} = K_{12} + U_s$$

We can also use conservation of momentum, as always. Using the same subscript labels as above,

$$p_{1i} + p_{2i} = p_{12}$$

Symbolic solution, "laboratory frame:" Initially, both blocks have kinetic energy, and the spring is uncompressed. At the moment of the spring's maximum compression, both blocks move together at the same speed, so we may treat them as a single block of mass $m_1 + m_2$ moving at velocity v. The spring will be compressed by an amount x at this moment, and hence stores potential energy $U_s = \frac{1}{2}kx^2$. Our energy balance is thus:

[i]Presumably, any energy losses due to destroying the blocks would occur just after this moment.

$$E_i = K_{1i} + K_{2i} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$
$$E_f = K_{12} + U_s = \frac{1}{2}(m_1+m_2)v^2 + \frac{1}{2}kx^2$$

Equating the initial and final energies, we see that the ΔK we desire is the same as the spring's potential energy.

$$\Delta K = K_{1i} + K_{2i} - K_{12} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}(m_1+m_2)v^2 = U_s = \frac{1}{2}kx^2$$

Once we find an expression for ΔK, we have the spring's compression. We can also apply conservation of momentum to this end:

$$\begin{aligned} p_{1i} + p_{2i} &= p_{12} \\ m_1v_1 + m_2v_2 &= (m_1+m_2)v \\ \Longrightarrow \quad v &= \frac{m_1v_1+m_2v_2}{m_1+m_2} = v_{com} \end{aligned}$$

We should not be surprised by this result …inserting our result for v into the our expression for ΔK eventually gives use the answer we seek.

$$\begin{aligned} \Delta K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}(m_1+m_2)v^2 \\ \Delta K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}(m_1+m_2)\left[\frac{m_1v_1+m_2v_2}{m_1+m_2}\right]^2 \\ 2\Delta K &= m_1v_1^2 + m_2v_2^2 - \frac{(m_1v_1+m_2v_2)^2}{m_1+m_2} \\ 2\Delta K &= \frac{(m_1+m_2)\left(m_1v_1^2+m_2v_2^2\right) - \left(m_1^2v_1^2 + 2m_1m_2v_1v_2 + m_2^2v_2^2\right)}{m_1+m_2} \\ 2\Delta K &= \frac{m_1^2v_1^2 + m_1m_2v_2^2 + m_1m_2v_1^2 + m_2^2v_2^2 - m_1^2v_1^2 - 2m_1m_2v_1v_2 - m_2^2v_2^2}{m_1+m_2} \\ 2\Delta K &= \frac{m_1m_2\left(v_1^2+v_2^2-2v_1v_2\right)}{m_1+m_2} = \frac{m_1m_2}{m_1+m_2}(v_1-v_2)^2 \\ \Longrightarrow \quad \Delta K &= \frac{1}{2}\mu v_{\text{rel}}^2 \end{aligned}$$

We can now easily find the spring's maximum compression:

$$\Delta K = \frac{1}{2}kx^2 \quad \Longrightarrow \quad x = \sqrt{\frac{\mu}{k}}v_{rel}$$

Symbolic solution, frame where block 2 is still: Here we imagine we are sitting on block 2 and watching the collision. This is a sensible frame to pick, since our desired result includes only the *relative* velocity anyway. In this frame of reference, the velocity of block 1 relative to block 2 is $v_1 - v_2 = v_{\text{rel}}$, and the velocity of block 2 is zero (since it is our reference point). The velocity of both blocks at the moment of maximum spring compression is then $v - v_2$, where v is the velocity of the two-block system with respect to the ground. Our kinetic energy balance is then

$$K_{1i} = \frac{1}{2} m_1 \left(v_1 - v_2\right)^2 \qquad K_{12} = \frac{1}{2}\left(m_1 + m_2\right)\left(v - v_2\right)^2$$

Momentum conservation is similarly straightforward, and gives us another expression for $v - v_2$

$$m\left(v_1 - v_2\right) = \left(m_1 + m_2\right)\left(v - v_2\right) \quad \Longrightarrow \quad v - v_2 = \frac{m_1}{m_1 + m_2}\left(v_1 - v_2\right)$$

Defining the kinetic energy change, and putting things together:

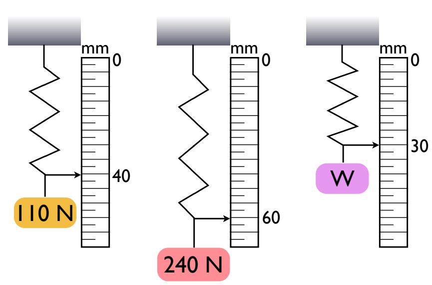
$$\begin{aligned}
\Delta K &= \frac{1}{2} m_1 \left(v_1 - v_2\right)^2 - \frac{1}{2}\left(m_1 + m_2\right)\left(v - v_2\right)^2 \\
\Delta K &= \frac{1}{2} m_1 \left(v_1 - v_2\right)^2 - \frac{1}{2}\left(m_1 + m_2\right)\left[\frac{m_1}{m_1 + m_2}\left(v_1 - v_2\right)\right]^2 \\
\Delta K &= \frac{1}{2}\left(v_1 - v_2\right)^2\left[m_1 - \frac{m_1^2}{m_1 + m_2}\right] = \frac{1}{2}\left(v_1 - v_2\right)^2\left[\frac{m_1^2 + m_1 m_2 - m_1^2}{m_1 + m_2}\right] \\
\Delta K &= \frac{1}{2}\left(v_1 - v_2\right)^2\left[\frac{m_1 m_2}{m_1 + m_2}\right] = \frac{1}{2}\mu v_{\text{rel}}^2
\end{aligned}$$

The maximum compression of the spring is found in the same way as above.

Numeric solution: Perhaps you have noticed we are not big on numbers.

Double check: For the first part, we were simply asked to show that the result is true . . . which seems to have worked out just fine. The second part relies only on conservation of energy and basic algebra. Qualitatively makes some sense that the spring compression is found by relating the kinetic energy change to the spring's potential energy. As the mass of either block increases, the reduced mass μ increases monotonically (since mass is always positive), and thus x increases, which is a sensible result.

3. A spring with a pointer attached is hanging next to a scale marked in millimeters. Three different packages are hung from the spring, in turn, as shown below. **(a)** Which mark on the scale will the pointer indicate when no package is hung from the spring? **(b)** What is the weight W of the third package?



Find: The weight of the unknown third package and equilibrium position of the spring.

Given: The position of the spring for two different known weights.

Sketch: The given sketch will be sufficient.

Relevant equations: We only need the weights of the first two packages and the force equation for a spring. Let the three packages have weights W_1, W_2 and W_3

$$W_1 = 110\,\mathrm{N} \quad W_2 = 240\,\mathrm{N} \quad W_3 = ?$$

The spring will respond with a force F when displaced a distance x from its equilibrium position x_{eq}

$$F = -k\left(x - x_{\text{eq}}\right)$$

Symbolic solution: The weights of a given package must equal the restoring force of the spring. When weight i is hung from the spring, it will stretch by an amount x_i from equilibrium:

$$W_i = k\left(x_i - x_{\text{eq}}\right)$$

The ratio of the weights of the first two packages gives us an equation with only x_{eq} as an unknown:

$$\begin{aligned}
\frac{W_1}{W_2} &= \frac{x_1 - x_{\text{eq}}}{x_2 - x_{\text{eq}}} \\
W_1 x_2 - W_1 x_{\text{eq}} &= W_2 x_1 - W_2 x_{\text{eq}} \\
(W_2 - W_1) &= W_2 x_1 - W_1 x_2 \\
\Longrightarrow \quad x_{\text{eq}} &= \frac{W_2 x_1 - W_1 x_2}{W_2 - W_1}
\end{aligned}$$

Given the equilibrium distance, subtracting the weight of the first two packages yields the spring constant, which we can use to find the weight of the third package:

$$\begin{aligned}
W_2 - W_1 &= kx_2 - kx_1 \\
k &= \frac{W_2 - W_1}{x_2 - x_1}
\end{aligned}$$

This makes sense – the distance the spring expands on changing the weight from W_1 to W_2 is $x_2 - x_1$, the force constant must be the ratio of this difference in force to the extra expansion distance. The weight of the third package is now readily found from the expressions for k and x_{eq}.

$$W_3 = k\left(x - x_{\text{eq}}\right) = \left(\frac{W_2 - W_1}{x_2 - x_1}\right)\left[x_3 - \left(\frac{W_2 x_1 - W_1 x_2}{W_2 - W_1}\right)\right]$$

Numeric solution: Given $W_1 = 110\,\mathrm{N}$ and $W_2 = 240\,\mathrm{N}$ along with $x_1 = 0.04\,\mathrm{m}$, $x_2 = 0.06\,\mathrm{m}$, and $x_3 = 0.03\,\mathrm{m}$,

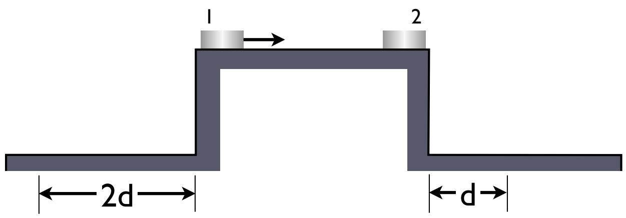
$$x_{\text{eq}} = \frac{W_2 x_1 - W_1 x_2}{W_2 - W_1} \approx 23\,\mathrm{mm}$$

$$W_3 = \left(\frac{W_2 - W_1}{x_2 - x_1}\right)\left[x_3 - \left(\frac{W_2 x_1 - W_1 x_2}{W_2 - W_1}\right)\right] \approx 45\,\mathrm{N}$$

Double check: You can verify easily that all of our expressions have the correct units. Clearly, package 3 must weigh less than either package 1 or 2, since it causes less expansion of the spring.

We can also approach this problem in a less formal manner, relying only on the fact that ideal springs have a linear force-displacement response. The difference in weight between packages 1 and 2 is 130 N and causes 0.02 m of extra expansion, meaning the spring should have a force constant of 6500 N/m. Package 3 stretches the spring by 0.01 m less than package 1, meaning it must weigh 65 N less than package 1, or 45 N. This is in the end exactly what our equations above tell us, we really only short-circuited the step of finding the equilibrium distance by subtracting displacements.

4. In the figure below, puck 1 of mass $m_1 = 0.20$ kg is sent sliding across a frictionless lab bench, to undergo a one-dimensional elastic collision with stationary puck 2. Puck 2 then slides off the bench and lands a distance d from the base of the bench. Puck 1 rebounds from the collision and slides off the opposite edge of the bench, landing a distance $2d$ from the base of the bench. What is the mass of puck 2?



We want the mass of the second puck. The two pucks undergo an elastic collision, with the second block initially at rest, and after the collision both pucks slide off the frictionless bench. The motion off of the bench is therefore projectile motion, with a purely horizontal velocity determined by the final velocities after the collision.

We can approach this by by conservation of energy. Initially, the total energy of the system is only the kinetic energy of the first puck, if we let the bench's surface be our zero for potential energy. After the collision, but before the pucks fall off of the bench, the total energy is the kinetic energy of both pucks. Between these two moments, the system's energy must be conserved, since the collision is elastic and there is no friction.

$$K_{1i} = \frac{1}{2}m_1 v_{1i}^2 = K_{1f} + K_{2f} = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_1 v_{2f}^2$$

After the collision, we also know that puck 2's speed is half that of puck 1, since it travels only half as far off the table!

$$2|v_{2f}| = |v_{1f}|$$

Thus,

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_1 v_{2f}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{8}m_1 v_{1i}^2 = \left(m_1 + \frac{1}{4}m_2\right) v_{1f}^2$$

$$\left(\frac{v_{1i}}{v_{1f}}\right)^2 = \frac{4m_1 + m_2}{4m_1}$$

Our collision equation also yields an expression for v_{1i}/v_{1f}:

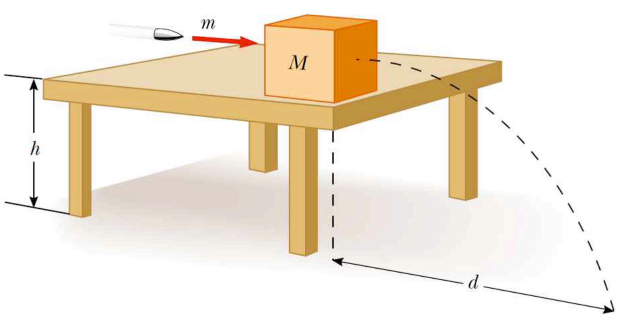
$$\frac{v_{1i}}{v_{1f}} = \frac{m_1 + m_2}{m_1 - m_2}$$

Thus,

$$\begin{aligned}
\frac{4m_1 + m_2}{4m_1} &= \left(\frac{m_1 + m_2}{m_1 - m_2}\right)^2 \\
\frac{4m_1 + m_2}{4m_1} &= \frac{m_1^2 + 2m_1m_2 + m_2^2}{m_1^2 - 2m_1m_2 + m_2^2} \\
4m_1^3 + 8m_1^2m_2 + 4m_1m_2^2 &= 4m_1^3 - 8m_1^2m_2 + 4m_1m_2^2 + m_1^2m_2 - 2m_1m_2^2 + m_2^3 \\
16m_1^2 &= m_2^2 + m_1^2 - 2m_1m_2 \qquad (m_2 \neq 0) \\
0 &= 15m_1^2 + 2m_1m_2 - m_2^2 \\
\implies \quad m_1 &= \frac{-2m_2 \pm \sqrt{4m_2^2 + 4m_2^2\,(15)}}{30} = \frac{-2m_2 \pm \sqrt{64m_2^2}}{30} = \left(\frac{-2 \pm 8}{30}\right) m_2 = \left\{\frac{1}{5}, -\frac{1}{3}\right\} m_2 \\
m_2 &= 5m_1 = 1\,\text{kg}
\end{aligned}$$

We have rejected the $m_2 < 0$ solution above as being silly.

5. A bullet of mass m is fired into a block of mass M initially at rest at the edge of a frictionless table of height h as in the figure below. The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet in terms of given quantities.



First we must handle the collision - which is clearly elastic since the bullet sticks in the block - and then we can use kinematics to find the initial velocity in terms of the masses and given distances. An inelastic collision clearly does not conserve energy, but it still conserves momentum. Thus, equating momentum before and after the collision (with the velocity of both objects just after the collision as v_f),

$$v_i m = (M + m)\, v_f \tag{1}$$

$$v_f = \left(\frac{m}{M + m}\right) v_i \tag{2}$$

If the block and bullet are launched off of the table horizontally, we know that the time t for it to reach the ground is the same for any other falling object. Given a starting height h, we know $h = \frac{1}{2}gt^2$. In this time t, it will travel a horizontal distance $d = v_f t$, so $t = d/v_f$. Putting these bits together, and substituting our expression for v_f:

$$h = \frac{1}{2}gt^2 = \frac{1}{2}g\frac{d^2}{v_f^2} = \frac{1}{2}gd^2\frac{(M+m)}{m^2}\frac{1}{v_i^2} \tag{3}$$

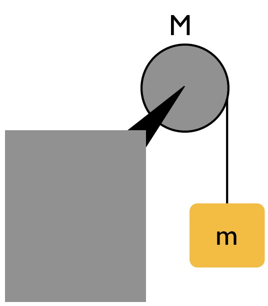
$$v_i^2 = \frac{gd^2}{2h}\left(\frac{M+m}{m}\right)^2 \tag{4}$$

$$v_i = d\left(\frac{M+m}{m}\right)\sqrt{\frac{g}{2h}} \tag{5}$$

We can check the units to see if our answer is plausible:

$$\left[\frac{m}{s}\right] = [m]\sqrt{\left[\frac{m}{s^2 m}\right]} = [m/s] \tag{6}$$

6. [From lecture] A uniform disk with mass $M = 2.5$ kg and radius $R = 20$ cm is mounted on a fixed horizontal axle, as shown below. A block of mass $m = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. *Note: the moment of inertia of a disk about its center of mass is* $I = \frac{1}{2}MR^2$.



We can first analyze the forces on the mass m. There is the weight of the mass pulling downward, and the tension T upward. Presuming the acceleration a to be downward,

$$\sum F = T - mg = -ma \tag{7}$$

$$T = m(g-a) \tag{8}$$

Next, we must deal with the pulley. The tension provides a force T pulling at a distance R from the center of rotation at a right angle, giving rise to a torque TR. Since the torque causes a clockwise rotation, it is by convention negative. Letting the moment of inertia of the pulley be kMR^2,

$$\sum \tau = -TR = I\alpha \tag{9}$$

If the rope does not slip on the pulley, then the linear acceleration of the rope must be the same as the linear acceleration of the pulley at the point it meets the rope, i.e., on the rim at distance R from the center of rotation. That means $a = R\alpha$, and now we have two expressions involving a, from which we may eliminate the tension:

$$
\begin{aligned}
I\alpha &= kMR^2\left(\frac{a}{R}\right) = -RT && (10)\\
T &= -kMa && (11)
\end{aligned}
$$

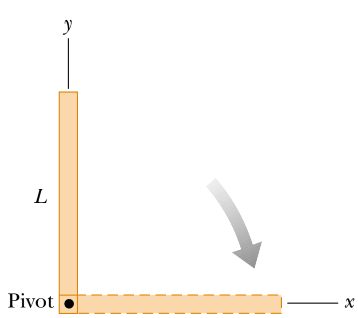
With $k = \frac{1}{2}$ for a disc, $T = -\frac{1}{2}Ma$. Now we can find the acceleration,

$$
\begin{aligned}
a &= \frac{T}{m} - g = -\frac{kMa}{m} - g && (12)\\
a\left(1 + \frac{kM}{m}\right) &= g && (13)\\
a &= \frac{g}{1 + \frac{kM}{m}} = \frac{mg}{m + kM} && (14)
\end{aligned}
$$

With $k = \frac{1}{2}$ for a disc,

$$
\begin{aligned}
a &= \frac{mg}{m + kM} = \frac{2mg}{2m + m} \approx -4.8\,\mathrm{m/s^2} && (15)\\
T &= -\frac{1}{2}Ma \approx -6\,\mathrm{N} && (16)\\
\alpha &= \frac{a}{R} \approx -24\,\mathrm{rad/s^2} && (17)
\end{aligned}
$$

7. A long uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin through one end. The rod is released from rest in a vertical position. At the instant the rod is horizontal, find its angular speed. The moment of inertia of a solid rod about its center of mass is $I = \frac{1}{12}ML^2$.



This is most easily approached by conservation of energy. The center of mass of the rod moves through a vertical distance $L/2$, so the rod's gravitational potential energy changes by $MgL/2$. This must be accompanied by a change in rotational kinetic energy of $\frac{1}{2}I\omega^2$. For a thin rod about its endpoint, $I = \frac{1}{3}ML^2$. Thus,

$$
\begin{aligned}
\Delta K &= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{1}{3}ML^2\omega^2 = -\Delta U = \frac{1}{2}MgL && (18)\\
\omega &= \sqrt{\frac{3g}{L}} && (19)
\end{aligned}
$$

One could then find the linear speed of the tip by noting that $v = \omega R$, and $R = L$ at the end of the rod.

8. [Note that rolling motion is not covered on exam 2.] A uniform ball of mass M and radius R rolls smoothly down a ramp at angle θ. The *center* of the ball starts at a vertical height h from the bottom of the ramp. How long does it take the ball to reach the bottom of the ramp?

If the center of the ball starts at a vertical height h, the center of mass of the ball moves through a height $h-R$. The distance traveled down the ramp you can verify by trigonometry is $d=\frac{h-R}{\sin\theta}$. The acceleration down the ramp will be constant, since it is only provided by gravity, so if we can find the speed at the bottom we can infer the time.

Conservation of energy gets us the speed. The rotational kinetic energy will be $\frac{1}{2}I\omega^2$ at the bottom. For rolling motion we know that $\omega=v/R$, and I is the moment of inertia about the center of mass, which we assume is kMR^2. Since the starting energy is purely potential and the final purely kinetic,

$$mg\left(h-R\right)=\frac{1}{2}I\omega^2+\frac{1}{2}Mv^2=\frac{1}{2}kMR^2\frac{v^2}{R^2}+\frac{1}{2}Mv^2 \tag{20}$$

$$v^2\left(M+kM\right)=2mg\left(h-R\right) \tag{21}$$

$$v=\sqrt{\frac{2mg\left(h-R\right)}{M\left(k+1\right)}} \tag{22}$$

With $k=\frac{2}{5}$ for a sphere,

$$v=\sqrt{\frac{10mg\left(h-R\right)}{7M}} \tag{23}$$

If the ball reached this velocity from rest under uniform acceleration, we know that $v_f^2=2ad$, or $a=v_f^2/2d$, and we also know $d=\frac{1}{2}at^2$. Thus,

$$t=\sqrt{\frac{2d}{a}}=\sqrt{\frac{4d^2}{v_f^2}}=\frac{2d}{v_f}=\frac{2\left(h-R\right)}{\sin\theta}\sqrt{\frac{m\left(k+1\right)}{2mg\left(h-R\right)}} \tag{24}$$

$$t=\sqrt{\frac{2\left(k+1\right)\left(h-R\right)}{g\sin^2\theta}}=\sqrt{\frac{14\left(h-R\right)}{5g\sin^2\theta}} \tag{25}$$