University of Alabama
Department of Physics and Astronomy

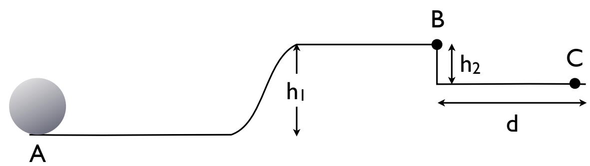
PH 105 / LeClair — Summer 2012

Rolling, Torque, Angular Momentum

1. Halliday, Resnick, & Walker, problem 11.14

Solution: Our sphere starts out at point A in the sketch below already undergoing smooth rolling motion, with center of mass velocity v_i. Since the sphere rolls without slipping, its angular and linear velocities must be related by the sphere's radius R, $v_i = R\omega$. We can apply conservation of mechanical energy to find the sphere's velocity at point B. Let the zero of gravitational potential energy be the lowest level in the diagram (the height of point A). At A, the total mechanical energy is purely kinetic, with both linear and rotational terms:

$$K_A + U_A = \frac{1}{2}mv_i + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_i + \frac{1}{2}I\frac{v_i^2}{R} = \frac{1}{2}v_i^2\left(m + \frac{I}{R^2}\right)$$



At point B, we also have translational and rotational kinetic energy, characterized by linear and angular velocities v_b and ω_b, respectively. We still have $v_b = R\omega_b$, since the motion is purely rolling without slipping. We also have now a gravitational potential energy mgh_1, and

$$K_B + U_B = \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1$$

Applying conservation of energy between A and B, we can solve for v_i:

$$\begin{aligned} K_A + U_A &= K_B + U_B \\ \frac{1}{2}v_i^2\left(m + \frac{I}{R^2}\right) &= \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1 \\ v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} \end{aligned}$$

We need only an expression for v_b. At point B, the sphere is launched from height h_2 above the far right platform, and it behaves just as any other projectile. In the absence of air resistance, the rate of rotation ω will not change from B to C, and we can therefore ignore the rotational motion. The sphere covers a horizontal distance d in a time t after being launched horizontally at v_b, and it covers a vertical distance h_2 in the same time t under the influence of gravity. Thus,

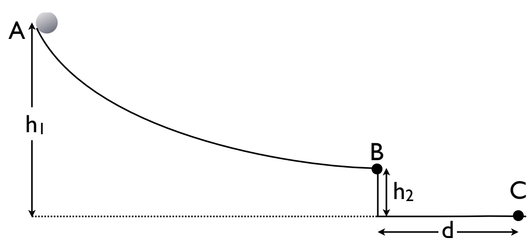
$$
\begin{aligned}
d &= v_b t \\
-h_2 &= -\frac{1}{2} g t^2 \\
\implies \quad v_b &= d\sqrt{\frac{g}{2h_2}}
\end{aligned}
$$

Using this result in our expression above, and noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$
\begin{aligned}
v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} = \frac{d^2 g}{2h_2} + \frac{2mgh_1}{m + I/R^2} \\
v_i^2 &= \frac{d^2 g}{2h_2} + \frac{2mgh_1}{m + \frac{2}{5}m} = \frac{d^2 g}{2h_2} + \frac{2gh_1}{\frac{7}{5}} = \frac{d^2 g}{2h_2} + \frac{10}{7} g h_1 \\
v_i &= \sqrt{\frac{d^2 g}{2h_2} + \frac{10}{7} g h_1} \approx 1.34\,\mathrm{m/s}
\end{aligned}
$$

2. Halliday, Resnick, & Walker, problem 11.16

Solution: First, a simple sketch for reference:



Once again, we need only apply conservation of energy. The object starts out at A with only gravitational potential energy, and at B has gained rotational and translational kinetic energy. Since we have rolling motion without slipping, we can relate linear and angular velocities at B via $v = R\omega$. Let the zero for gravitational potential energy by the lowest level in the figure (that of C). Conservation of energy between A and B yields:

$$
\begin{aligned}
mgH &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh \\
mg\left(H - h\right) &= \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = \frac{1}{2}v^2\left(m + \frac{I}{R^2}\right) = \frac{1}{2}v^2\left(m + \beta m\right) = \frac{1}{2}mv^2\left(1 + \beta\right) \\
1 + \beta &= \frac{2g\left(H - h\right)}{v^2} \\
\beta &= \frac{2g\left(H - h\right)}{v^2} - 1
\end{aligned}
$$

We need only an expression for v. Just as in the previous problem, we can use the equations of projectile motion.

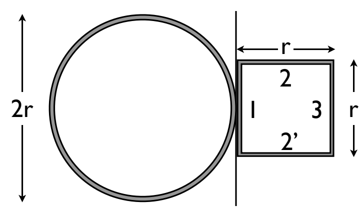
$$d = vt$$
$$-h = -\frac{1}{2}gt^2$$
$$\implies \quad v = d\sqrt{\frac{g}{2h}}$$

Thus,

$$\beta = \frac{2g\,(H-h)}{v^2} - 1 = \frac{2g\,(H-h)}{\frac{d^2 g}{2h}} - 1 = \frac{4h\,(H-h)}{d^2} - 1 \approx 0.25$$

3. Halliday, Resnick, & Walker, problem 11.41

Solution: Again, a quick sketch:



The square is made up of four thin rods of length r, while the hoop has radius r. First, we calculate the moment of inertia of the square. The first rod labeled "1" is on the axis of rotation. If its thickness is negligible, its moment of inertia is essentially zero – all the mass is at distance zero from the axis of rotation. The horizontal rods 2 and 2′ are both rotating about a distance $r/2$ from their center of mass, and thus

$$I_2 = I_{2'} = I_{com} + m\left(\frac{r}{2}\right)^2 = \frac{1}{12}mr^2 + \frac{1}{4}mr^2 = \frac{1}{3}mr^2$$

The rod labeled 3 has all its mass located a distance r from the axis of rotation (still presuming the thickness to be negligible), and thus its moment of inertia is the same as that of a particle of mass m a distance r from the axis of rotation, $I_3 = mr^2$. In total,

$$I_{\square} = I_1 + I_2 + I_{2'} + I_3 = 0 + \frac{1}{3}mr^2 + \frac{1}{3}mr^2 + mr^2 = \frac{5}{3}mr^2$$

Our hoop rotates a distance r from its center of mass, and thus

$$I_{\circ} = I_{com} + mr^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

The total system then has

$$I_{\text{tot}} = I_{\square} + I_{\circ} = \left(\frac{5}{3} + \frac{3}{2}\right)mr^2 = \frac{19}{6}mr^2 \approx 1.6\,\text{kg}\,\text{m}^2$$

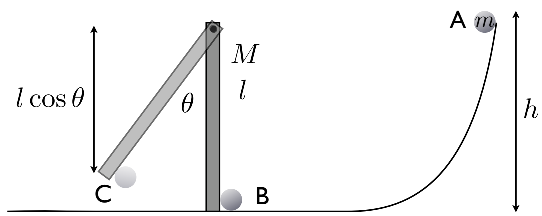
The total angular momentum can be found from the moment of inertia and the angular velocity, the latter of which can be found easily from the period of rotation:

$$
\omega = \frac{2\pi}{T}
$$

$$
L = I_{\text{tot}}\omega = \frac{2\pi I_{\text{tot}}}{T} = \frac{19\pi m r^2}{3T} \approx 4.0\,\text{kg}\,\text{m}^2/\text{s}
$$

4. Halliday, Resnick, & Walker, problem 11.66

Solution: Again, a quick sketch. Let A be the starting point, B the moment of collision between the ball and rod, and C the point when maximum height is reached by the rod + ball system. We approximate the ball as a point mass, since we are told it is small (and we anyway have no way of calculating its moment of inertia, since we do not have any geometrical details ...).



The velocity v of the ball at point B can be found using conservation of mechanical energy. Let the floor be the height of zero gravitational potential energy.

$$
\begin{aligned}
K_A + U_A &= K_B + U_B \\
mgh &= \frac{1}{2}mv^2 \\
\implies v &= \sqrt{2gh}
\end{aligned}
$$

The collision is clearly inelastic, since the ball sticks to the rod. We could use conservation of linear momentum, but this would require breaking up the rod into infinitesimal discrete bits of mass and integrating over its length. Easier is to use conservation of *angular* momentum about the pivot point of the rod. Just before the collision, we have the ball moving at speed v a distance l. Let $\hat{\boldsymbol{\imath}}$ be to the right, and $\hat{\boldsymbol{\jmath}}$ upward (making $\hat{\boldsymbol{k}}$ into the page). The initial angular momentum is then

$$
\vec{L}_i = \vec{r} \times \vec{p} = l\,\hat{\boldsymbol{\jmath}} \times (-mv\,\hat{\boldsymbol{\imath}}) = -mvl\,(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}) = mvl\,\hat{\boldsymbol{k}} = ml\sqrt{2gh}\,\hat{\boldsymbol{k}}
$$

After the collision, we have the rod and mass stuck together, rotating at angular velocity ω. Defining counterclockwise rotation to be positive as usual, the final angular momentum is thus

$$
\vec{L}_f = I\omega\,\hat{\boldsymbol{k}}
$$

The total moment of inertia about the pivot point is that of the rod rotating plus that of the ball. The rod rotates a distance $l/2$ from its center of mass, and again we approximate the ball as a point mass rotating at a distance l (since we told it is small).

$$I = I_{\text{rod}} + I_{\text{ball}} = I_{\text{rod, com}} + M\left(\frac{l}{2}\right)^2 + ml^2 = \frac{1}{12}Ml^2 + Ml^2 + ml^2 = \left(\frac{1}{3}M + m\right)l^2$$

Equating initial and final angular momentum, we can solve for the angular velocity after the collision:

$$L_f = I\omega = L_i = mvl = ml\sqrt{2gh}$$
$$\left(\frac{1}{3}M + m\right)l^2\omega = ml\sqrt{2gh}$$
$$\omega = \frac{m\sqrt{2gh}}{\left(\frac{1}{3}M + m\right)l}$$

At this point, we may use conservation of energy once again. When the system reaches its maximum angle θ at C, the center of mass of the rod + ball system will have moved up by an amount Δy_{cm}. The change in gravitational potential energy related to this change in center of mass height must be equal to the rotational kinetic energy just after the collision. Thus,

$$\frac{1}{2}I\omega^2 = \frac{\vec{L}\cdot\vec{L}}{2I} = \frac{L^2}{2I} = (m + M)\,g\Delta y_{cm}$$

Here we have noted that the rotational kinetic energy can be related to the angular momentum to save a bit of algebra. To proceed, we must find the difference in the center of mass height between points C and B. Let $y\!=\!0$ be the height of the floor. At point B,

$$y_{cm,B} = \frac{M\left(\frac{L}{2}\right) + m\,(0)}{m + M} = \left(\frac{l}{2}\right)\left(\frac{M}{m + M}\right)$$

At point C, the ball is now at a height $l - l\cos\theta$, while the center of mass of the rod (its midpoint) is now at $l - l\cos\theta + \frac{1}{2}l\cos\theta$. Thus,

$$y_{cm,C} = \frac{M\left(l - l\cos\theta + \frac{1}{2}l\cos\theta\right) + m\,(l - l\cos\theta)}{m + M} = \frac{Ml\left(1 - \frac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta)}{m + M}$$

The change in center of mass height can now be found:

$$\begin{aligned}\Delta y_{cm} = y_{cm,C} - y_{cm,B} &= \frac{Ml\left(1 - \frac{1}{2}\cos\theta\right) + ml\,(1 - \cos\theta) - \frac{1}{2}Ml}{m + M}\\ &= \frac{\frac{1}{2}Ml\,(1 - \cos\theta) + ml\,(1 - \cos\theta)}{m + M}\\ &= \frac{l}{m + M}\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right)\end{aligned}$$

Using our previous energy balance between B and C,

$$\frac{L^2}{2I} = (m + M)\,g\Delta y_{cm} = lg\,(1 - \cos\theta)\left(m + \frac{1}{2}M\right)$$

Since the initial and final angular momenta are equal, we may substitute either L_f or L_i, the latter being the easiest option. This is not strictly *necessary* – we could use L_f or even just grind through $\frac{1}{2}I\omega^2$ and the result must be the same. However, using L_i here saves quite a bit of algebra in the end when we try to put θ in terms of only given quantities. Doing so, and solving for θ

$$\frac{L_f^2}{2I} = \frac{L_i^2}{2I} = \frac{2l^2m^2gh}{2\left(\frac{1}{3}M+m\right)l^2} = lg\left(1-\cos\theta\right)\left(m+\frac{1}{2}M\right)$$

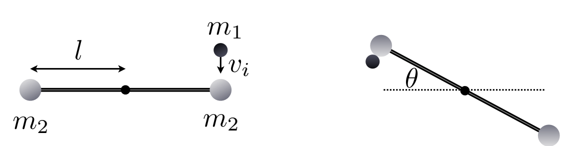
$$1-\cos\theta = \frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}$$

$$\theta = \cos^{-1}\left[1-\frac{m^2h}{l\left(\frac{1}{3}M+m\right)\left(\frac{1}{2}M+m\right)}\right] \approx 32^\circ$$

Note that for $m=0$, $\theta=0$, as we expect. On the other hand, for $M=0$ we have $\cos\theta = 1-h/l = 1/2$. This means that the particle is at a height $l - l\cos\theta = l/2 = h$ at point C – exactly what we would expect if mechanical energy were conserved!

5. Halliday, Resnick, & Walker, problem 11.65

Solution: A quick sketch.



(a) Our dumbbell, consisting of two masses m_2 both a distance l from its center of mass, is struck by a smaller mass m_1 traveling at velocity $\vec{v}_i$. Conservation of angular momentum can be used to find the angular velocity after the collision. Before the collision, with $\hat{\imath}$ to the right and $\hat{\jmath}$ upward, we have the smaller mass' momentum $\vec{p}_i = -m_1v_i\,\hat{\jmath}$ acting at a distance $\vec{r} = l\,\hat{\imath}$ from the center of rotation.

$$\vec{L}_i = \vec{r}\times\vec{p} = -m_1lv_i\,\hat{\boldsymbol{k}}$$

The minus sign indicates a clockwise rotation following our usual convention, which is sensible. After the collision, the entire system rotates clockwise at angular velocity $\vec{\omega} = -\omega\,\hat{\boldsymbol{k}}$. The total moment of inertia is found easily, since we have only point-like masses:

$$I = \sum_i m_ir_i^2 = m_2l^2 + m_2l^2 + m_1l^2 = l^2\left(2m_2+m_1\right)$$

The final angular momentum is then

$$\vec{L}_f = I\vec{\omega} = -l^2\omega\left(2m_2+m_1\right)\,\hat{\boldsymbol{k}}$$

Conservation of angular momentum gives us

$$\vec{\omega} = \frac{L_i}{I} = \frac{m_1 v_i}{(2m_2 + m_1)\, l}\, \hat{\boldsymbol{k}} \approx 0.15\,\text{rad/s}\,\hat{\boldsymbol{k}}$$

(b) The initial kinetic energy of the system is only that of the smaller mass, $K_i = \frac{1}{2} m_1 v_i^2$. The final kinetic energy is the rotational kinetic energy of the whole system, which is simplified a bit in terms of angular momentum

$$K_f = \frac{1}{2} I\omega^2 = \frac{\vec{L}\cdot\vec{L}}{2I} = \frac{L_i^2}{2I} = \frac{m_1^2 l^2 v_i^2}{2l^2\,(2m_2 + m_1)} = \frac{1}{2} m_1 v_i^2 \left(\frac{m_1}{2m_2 + m_1}\right) = K_i \left(\frac{m_1}{2m_2 + m_1}\right)$$

Note that since angular momentum is conserved, we can use either L_i or L_f in the kinetic energy equation; using L_i is somewhat simpler algebraically. The ratio of final to initial kinetic energies is thus

$$\frac{K_f}{K_i} = \frac{m_1}{2m_2 + m_1} \approx 0.0123$$

(c) What happens once the system starts rotating? Even without the initial kinetic energy of the smaller mass, since all forces present after the collision are conservative the whole system would have enough energy to rotate through 180°, since that would put all of the masses back at the same height. The gravitational potential energy of the system right after the collision is the same as that after rotating through 180°, so the system must rotate at least that much.

After rotating through 180°, the total mechanical energy of the system is unchanged from the point right after the collision. The system will continue rotating through a further maximum angle θ at which point the gain in potential energy equals the kinetic energy right after the collision, K_f. As the system rotates, one of the m_2 masses will go up by an amount $h = l\sin\theta$, and the other m_2 mass will go down by the same amount. The only change in potential energy comes from the smaller m_1 mass moving up by h! We can balance mechanical energy between configurations right after the collision, after rotating through 180°, and after rotating through an additional angle θ. Let the initial horizontal axis of the dumbbell be the zero of potential energy.

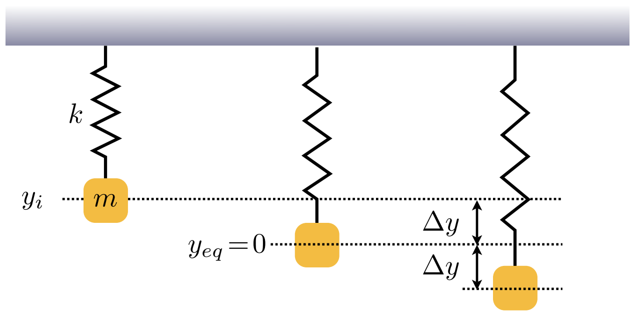
$$\begin{aligned}
\text{after collision:} \quad & K + U = K_f \\
\text{after rotating through } 180^\circ\text{:} \quad & K + U = K_f \\
\text{after an additional rotation by } \theta\text{:} \quad & K + U = m_2 g l \sin\theta + m_1 g l \sin\theta - m_2 g l \sin\theta = m_1 g l \sin\theta \\
\text{conservation of mechanical energy} \quad \Longrightarrow \quad & m_1 g l \sin\theta = K_f = \frac{m_1^2 v_i^2}{2\,(2m_2 + m_1)} \\
& \sin\theta = \frac{m_1 v_i^2}{2gl\,(2m_2 + m_1)} \\
& \theta = \sin^{-1}\left[\frac{m_1 v_i^2}{2gl\,(2m_2 + m_1)}\right] \approx 1.3^\circ
\end{aligned}$$

The total angle of rotation is thus $180^\circ + 1.3^\circ = 181.3^\circ$.

Oscillations

6. Halliday, Resnick & Walker Problem 15.37: mass hanging from a spring.

Our mass starts out at position y_i, corresponding to the un-stretched length of the spring. When released, its lowest position is $2\Delta y = 10\,\text{cm}$ below y_i during the subsequent oscillations. This means that the amplitude of the simple harmonic motion is Δy, symmetric about an equilibrium position y_{eq} – both y_i and the lowest point in the motion are Δy from y_{eq}. For convenience, let the equilibrium position be our origin, $y_{eq} = 0$, with the $\hat{\boldsymbol{\jmath}}$ direction being upward. With this choice, $y_i = \Delta y$ is the amplitude of harmonic motion. Make use of the figure below.



(a) We can find the frequency of oscillation by considering the forces acting on the mass, which are only gravity and the spring restoring force. If the mass moves a distance y from equilibrium,

$$ma = mg - ky$$

At the equilibrium position, the string is stretched by an amount Δy compared to its natural length, and $a = 0$:

$$mg = k\Delta y \quad \Longrightarrow \quad \frac{k}{m} = \frac{g}{\Delta y}$$

In principle, we can now must use $f = (2\pi)^{-1}\sqrt{k/m}$ to find the frequency of oscillation. However, should we be concerned whether our solution to simple harmonic motion is valid in the presence of an additional constant force (i.e., gravity)? Our force balance equation, suitably rearranged, reads

$$\frac{d^2y}{dt^2} + \frac{k}{m}y - g = 0$$

Without the additional constant gravitational acceleration, we would have our equation for simple harmonic motion. A simple substitution will recover the usual equation for simple harmonic motion, for which we know the solution. Let $y' = y - mg/k$, which gives $d^2y'/dt^2 = d^2y/dt^2$. Making the substitution in our equation above,

$$\frac{d^2y'}{dt^2} + \frac{k}{m}y' + g - g = \frac{d^2y'}{dt^2} + \frac{k}{m}y' = 0$$

We have recovered the standard equation of motion for a simple harmonic oscillator, and thus the presence of an additional constant force serves only to shift the origin by an amount mg/k. This shift leaves the frequency of oscillation unchanged at $f = (2\pi)^{-1}\sqrt{k/m}$. The substitution we made physically corresponds to shifting the equilibrium position downward by an amount mg/k – exactly how far the mass pulls the spring down once it is attached. This shift is just a choice of origin so far as the equations are concerned, the physics is unchanged. In the end, we are justified in using our beloved equations of simple harmonic motion, so long as we choose our origin at the new equilibrium position $y_i - mg/k$, which we have already done!

With our now-justified solution the numbers given,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta y}} \approx 2.2\,\text{Hz}$$

Recall that units of s^{-1} are commonly called *Hertz*, abbreviated Hz.

(b) When the mass is 8 cm below its initial position, what is its speed? There are several ways to go about this.

Conservation of Energy: First, and perhaps most straightforwardly, we can use conservation of mechanical energy. Let the position of interest at 8 cm be y_f. At the starting position of the mass, y_i, we have only the gravitational potential energy of the mass, since the mass is at rest and the spring is un-stretched. At position y_f, the mechanical energy consists of three parts: the new gravitational potential energy, the kinetic energy of the mass, and the potential energy of the now stretched spring. For the latter term, it is key to remember that the spring has been stretched by an amount $y_i - y_f$, since it started at its un-stretched length at y_i.[i] Writing down all the requisite energy terms, it is no big trick to solve for v

$$\begin{aligned}
mgy_I &= mgy_f + \frac{1}{2}mv^2 + \frac{1}{2}k\left(y_i - y_f\right)^2 \\
\frac{1}{2}mv^2 &= mg\left(y_i - y_f\right) - \frac{1}{2}k\left(y_i - y_f\right)^2 \\
v^2 &= 2g\left(y_i - y_f\right) - \frac{k}{m}\left(y_i - y_f\right)^2 \qquad \left(\text{note} \quad \frac{k}{m} = \frac{g}{\Delta y}\right) \\
v &= \sqrt{2g\left(y_i - y_f\right) + \frac{g}{\Delta y}\left(y_i - y_f\right)^2}
\end{aligned}$$

Noting that we are told $y_i - y_f = 8\,\text{cm}$ and $\Delta y = 5\,\text{cm}$ (and converting everything to meters),

$$v = \pm 0.56\,\text{m/s}$$

The $\pm$ in this case is physically meaningful – at 8 cm below the starting position, the mass can be going either upward or downward with the same speed.

Equation of Motion: Since we have established that our hanging mass follows simple harmonic motion, we know the general solution for $y(t)$:

$$y(t) = A\cos\omega t + B\sin\omega t$$

[i] Be careful that in the present case the equilibrium position is *not* the un-stretched position, and therefore not the position of zero spring potential energy.

From $y(t)$, we can readily find $v = dy/dt$, we need only find the time at which $y(t)$ corresponds to the given position. If the mass starts at rest at y_i, our boundary conditions are $y(0) = 0$ and $v(0) = 0$. Our general solution then becomes

$$y(t) = y_i \cos \omega t$$

in order to be consistent with our boundary conditions. At what time does $y(t)$ correspond to the point of interest? The mass starts out at $y_i = 5\,\text{cm}$ above equilibrium. That means that the position of interest, $8\,\text{cm}$ below y_i, is then $3\,\text{cm}$ *below* equilibrium. Thus, we are interested in the time t_o such that $y(t_o) = -3 \equiv y_f$ (since $\hat{\boldsymbol{\jmath}}$ is upward).

$$\begin{aligned} y_f &= y_i \cos \omega t_o \\ t_o &= \frac{1}{\omega} \cos^{-1} \left[\frac{y_f}{y_i} \right] \end{aligned}$$

The velocity is now easily found:

$$\begin{aligned} v(t) &= \frac{dy}{dt} = -\omega y_i \sin \omega t \\ v(t_o) &= -\omega y_i \sin \left[\cos^{-1} \left(\frac{y_f}{y_i} \right) \right] = -\omega y_i \sqrt{1 - \left(\frac{y_f}{y_i} \right)^2} = -\omega \sqrt{y_i^2 - y_f^2} \\ &= -2\pi f \sqrt{y_i^2 - y_f^2} \approx 0.56\,\text{m/s} \end{aligned}$$

Note that we used the identity $\sin\left[\cos^{-1} x\right] = \sqrt{1 - x^2}$ here. Also note that this is simply the equation of an ellipse, which leads us to our next method ...

Phase space relationships: As we discussed in class, for the general simple harmonic motion solution

$$y(t) = C \cos(\omega t + \delta)$$

The allowed values of position y and momentum p for our oscillator satisfy the equation of an ellipse:

$$\frac{y^2}{C^2} + \frac{p^2}{m^2 \omega^2 C^2} = 1$$

That is, position and momentum are conjugate variables, and their values are linked. Since we know the position of interest y, there are at most two possible momenta, which will differ only by a sign. Noting that in the present case our boundary conditions give $C = y_i$, and using $p = mv$

$$\begin{aligned} 1 - \frac{y^2}{y_i^2} &= \frac{m^2 v^2}{m^2 \omega^2 y_i^2} = \frac{v^2}{\omega^2 y_i^2} \\ v^2 &= \omega^2 y_i^2 \left(1 - \frac{y^2}{y_i^2} \right) = \omega^2 \left(y_i^2 - y_f^2 \right) \\ v &= \pm \omega \sqrt{y_i^2 - y_f^2} = \pm 2\pi f \sqrt{y_i^2 - y_f^2} \end{aligned}$$

Precisely the same solution, quite a bit faster.

(c) We are told that the addition of a 0.3 kg mass halves the frequency of oscillation. If the original mass is m_1, and the new mass is $m_2 = 0.3\,\text{kg}$, the original frequency is

$$f_o = \sqrt{\frac{k}{m_1}}$$

The new frequency is determined by the total mass, now $m_1 + m_2$:

$$f = \frac{1}{2} f_o = \sqrt{\frac{k}{m_1 + m_2}}$$

Combining, and solving for m_1,

$$\begin{aligned} \frac{1}{2} f_o = \frac{1}{2}\sqrt{\frac{k}{m_1}} &= \sqrt{\frac{k}{m_1 + m_2}} \\ \frac{k}{4m_1} &= \frac{k}{m_1 + m_2} \\ k\left(m_1 + m_2\right) &= 4km_1 \\ 3m_1 &= m_2 \quad \Longrightarrow \quad m_1 = 0.1\,\text{kg} \end{aligned}$$

(d) The new equilibrium position is found just like the original equilibrium position: the total weight balances the spring's restoring force. Let the new equilibrium position be a distance y'_{eq} below the original equilibrium:

$$\begin{aligned} ky'_{eq} &= \left(m_1 + m_2\right) g \\ y'_{eq} &= \frac{g}{k}\left(m_1 + m_2\right) = \frac{g\Delta y}{m_1 g}\left(m_1 + m_2\right) \qquad \left(\text{note} \quad k = \frac{m_1 g}{\Delta y}\right) \\ &= \left(\frac{m_1 + m_2}{m_1}\right)\Delta y = 4\Delta y \qquad (\text{note} \quad 3m_1 = m_2) \\ &\approx 0.2\,\text{m} \end{aligned}$$

7. Halliday, Resnick & Walker Problem 15.55 (HW 11)

In the end, we only have a physical pendulum, and the period is given by

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

where I is the moment of inertia of the rod (of mass m) about the pivot point, and h is the distance between the rod's center of mass and the pivot point. Let the pivot be a distance x from the end of the rod, making it a distance $l/2 - x$ from the center of mass. The moment of inertia is then

$$I = I_{com} + m\left(\frac{l}{2} - x\right)^2 = \frac{1}{12}ml^2 + m\left(\frac{l}{2} - x\right)^2$$

The distance between the center of mass and the pivot is $h = l/2 - x$, so

$$I = \frac{1}{12}ml^2 + mh^2$$

The period is thus

$$T = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}} = 2\pi\sqrt{\frac{l^2}{12gh} + \frac{h}{g}}$$

We wish to find x such that T is a maximum, which means $dT/dx = 0$. Noting that $dT/dx = -dT/dh$,

$$\begin{aligned}
\frac{dT}{dx} &= -\frac{dT}{dh} = 0 \\
\frac{d}{dh}\left[2\pi\sqrt{\frac{\frac{1}{12}l^2 + h^2}{gh}}\right] &= 0 \\
2\pi\left(\frac{1}{2}\right)\left(\frac{-l^2}{12gh^2} + \frac{1}{g}\right)\left(\frac{\frac{1}{12}l^2 + h^2}{gh}\right)^{-1/2} &= 0 \\
\implies \quad \frac{-l^2}{12gh^2} + \frac{1}{g} &= 0 \\
12h^2 &= l^2 \\
h &= \frac{l}{2\sqrt{3}} \approx 0.29l
\end{aligned}$$

A quick second derivative test or a plot of dT/dh verifies that this is indeed a minimum, not a maximum. The minimum period is therefore

$$T_{\min} = T\bigg|_{h=\frac{l}{2\sqrt{3}}} = 2\pi\sqrt{\frac{\frac{1}{12}l^2 + \frac{1}{12}l^2}{g\frac{l}{2\sqrt{3}}}} = 2\pi\sqrt{\frac{l}{\sqrt{3}g}} \approx 2.26\,\mathrm{s}$$

(b) Given $T \propto \sqrt{l}$, the period increases as l increases.

(c) The period is independent of m, and remains unchanged as m increases.

8. Halliday, Resnick & Walker Problem 15.100

The mechanical energy in this case consists of rotational kinetic energy, translational kinetic energy, and potential energy stored in the spring. Let $x = 0$ correspond to the un-stretched length of the spring, which is also the equilibrium position of this system. The total mechanical energy is

$$E_{tot} = K + U = K_t + K_r + U_s = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2$$

Before we proceed, one aside: if a circular object is rolling smoothly, the frictional force plays no roll - essentially, at each instant in time it is a different bit of the circular surface contacting the ground. Friction only imparts a retarding force and dissipates energy from the system when there is sliding involved. See `http://webphysics.davidson.edu/faculty/dmb/py430/friction/rolling.html` for a good explanation. Basically, pure rolling involves no work done by friction, so we are justified in writing the total energy as we have above.

In the case of pure rolling, we can relate the linear velocity of the center of mass v and the angular velocity ω through the radius of the cylinder, $v = r\omega$. Substituting for ω above, and noting $I = kmr^2$ in general (with $k = 1/2$ in the present case)

$$\begin{aligned} E_{tot} &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\left(kmr^2\right)\left(\frac{v}{r}\right)^2 + \frac{1}{2}kx^2 \\ &= \left(\frac{k+1}{2}\right)mv^2 + \frac{1}{2}kx^2 \\ &= (k+1)\,K_t + \frac{1}{2}kx^2 \end{aligned}$$

We are told the maximum displacement is $x_{\text{max}} = \frac{1}{4}$ m. At maximum displacement, both kinetic terms are zero, and the energy is purely potential:

$$\frac{1}{2}kx_{\text{max}}^2 = E_{\text{tot}} = \frac{3}{32}\text{ J}$$

On the other hand, at the equilibrium position, the energy is entirely kinetic. Since the only relevant forces are conservative (having established friction plays no role), mechanical energy is conserved, at equilibrium

$$\begin{aligned} K_r + K_t &= (k+1)\,K_t = E_{\text{tot}} \qquad \left(k = \frac{1}{2}\right) \\ \Longrightarrow K_t &= \frac{E_{\text{tot}}}{k+1} = \frac{1}{16}\text{ J} \\ K_r &= E_{\text{tot}} - K_t = \frac{1}{32}\text{ J} \end{aligned}$$

In order to find the period of motion, we would like to find $a = d^2x/dt^2$ and show that it is proportional to position, $a = -\omega^2 x$. We could write down a force and torque balance and arrive at the solution without an inordinate amount of work. However, there is an easier way.

We can also find the period by noting that $dE/dT = 0$, since mechanical energy is conserved. Taking the time derivative of the total energy will give us factors of acceleration and position; if we are lucky, that is all.

$$\begin{aligned} \frac{dE_{tot}}{dt} &= \frac{d}{dt}\left[\left(\frac{k+1}{2}\right)mv^2 + \frac{1}{2}kx^2\right] = 0 \\ 0 &= \left(\frac{k+1}{2}\right)m\,(2v)\left(\frac{dv}{dt}\right) + \frac{1}{2}k\,(2x)\left(\frac{dx}{dt}\right) \\ 0 &= (k+1)\,mva + kxv \\ 0 &= (k+1)\,ma + kx \qquad (v \neq 0) \\ a &= -\frac{k}{m\,(k+1)}x \equiv -\omega^2 x \end{aligned}$$

This is just the usual equation for simple harmonic motion, for which we know the solution

$$\omega = \sqrt{\frac{k}{m\,(k+1)}}$$
$$T = \frac{2\pi}{\omega}$$

Using $k=1/2$,

$$\omega = \sqrt{\frac{2k}{3m}}$$
$$T = 2\pi\sqrt{\frac{3m}{2k}}$$

The division by v above means that this solution is not valid at the turning points, where $v=0$, which is not really a restriction at all.

9. Halliday, Resnick & Walker Problem 15.26. Two blocks ($m = 1.8\,\text{kg}$ and $M = 10\,\text{kg}$) and a spring ($k = 200\,\text{N/m}$) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the the spring-blocks system puts the smaller block on the verge of slipping over the larger block?

If the upper block m is on the verge of slipping, it means that the force exerted on it by the larger block equals the maximum force of static friction $f_{s,\text{max}}=\mu_s mg$. The force exerted on the smaller block is due to the acceleration of the larger block, which we know to be $a=\omega^2 x_m$ during simple harmonic motion. If the acceleration a exceeds $f_{s,\text{max}}/m$, the smaller block will fall off. The angular frequency of simple harmonic motion ω is readily found by noting that the spring k is connected to a total mass $M+m$

$$\omega=\sqrt{\frac{k}{M+m}}$$

The amplitude of simple harmonic motion gives us a maximal acceleration, which we can compare with $f_{s,\text{max}}/m$ - the latter must be larger to avoid the smaller block falling off.

$$\frac{f_{s,\text{max}}}{m} > \omega^2 x_m$$
$$\mu_s g > \frac{k x_m}{M+m}$$
$$x_m < \frac{\mu_s g\,(M+m)}{k} \approx 0.23\,\text{m}$$

10. Halliday, Resnick & Walker Problem 15.56. A 2.50 kg disk of diameter $D=42.0\,\text{cm}$ is supported by a rod of length $L=76.0\,\text{cm}$ and negligible mass that is pivoted at its end. **(a)**With the massless torsion spring unconnected, what is the period of oscillation? **(b)** With the torsion spring connected, the rod is vertical at equilibrium. What is the torsion constant of the spring if the period of oscillation has been decreased by 0.500 s?

(a) Without the torsion spring, this is just a physical pendulum. In order to find its period, we need only the moment of inertia of the pendulum bob about the pivot point of the pendulum. The bob is a simple disk, with $I_{com} = \frac{1}{2}mr^2$, and it rotates about the pendulum's pivot point, a distance $r + L$ away (here $r = D/2$ and m is the mass of the disk). The moment of inertia is then found from the parallel axis theorem:

$$I = \frac{1}{2}mr^2 + m\left(r + L\right)^2$$

We have already derived the formula for the period of a physical pendulum (T_o), given the moment of inertia I and the distance from the bob's center of mass to the pivot point h:

$$T_o = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{2}mr^2 + m\left(r + L\right)^2}{mg\left(r + L\right)}} = 2\pi\sqrt{\frac{\frac{1}{2}r^2 + \left(r + L\right)^2}{g\left(r + L\right)}} \approx 2.00\,\text{s}$$

(b) If the pendulum has a shorter period when the torsion spring is connected, this must mean that the restoring torque due to the spring acts in the same direction as gravity. We can find the period of the new pendulum by considering both torques together, and noting that the sum of all torques must give I times the angular acceleration.

If our pendulum is inclined at an angle θ relative to its vertical equilibrium position, the magnitude of the torque due to a torsion spring is $\kappa\theta$, while the torque on the pendulum due to the weight of the bob is $mgh\sin\theta$. The torque balance then reads (noting that we have "restoring torques" present to get the signs right)

$$\sum\tau = -mgh\sin\theta - \kappa\theta = I\frac{d^2\theta}{dt^2}$$

If we assume small deviations from equilibrium (small θ), then $\sin\theta \approx \theta$, we recover our equation for simple harmonic motion:

$$\begin{aligned} -mgh\sin\theta - \kappa\theta \approx -\left(mgh + \kappa\right)\theta &= I\frac{d^2\theta}{dt^2} \\ \frac{d^2\theta}{dt^2} &= -\left(\frac{mgh + \kappa}{I}\right)\theta \\ \Longrightarrow \qquad \omega &= \sqrt{\frac{mgh + \kappa}{I}} \\ T &= 2\pi\sqrt{\frac{I}{mgh + \kappa}} \end{aligned}$$

We know the new period T with the torsion spring is 0.500 sec shorter than T_o, $T - T_o = -0.500\,s$, so we know enough to find κ

$$T = 2\pi\sqrt{\frac{I}{mgh+\kappa}}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{mgh+\kappa}{I}$$

$$\kappa = I\left(\frac{2\pi}{T}\right)^2 - mgh = I\left(\frac{2\pi}{T_o - 0.500}\right)^2 - mgh \approx 18.4\,\mathrm{N\,m/rad}$$

Waves

11. Halliday, Resnick & Walker Problem 16.25. A uniform rope of mass m and length L hangs from a ceiling.

(a) The speed of transverse wave as a function of the position y from the bottom of the rope depends only on the tension at that point. You can't push on a rope, so the tension at that point depends only on how much rope is hanging below that point, a length y. If we define a linear density $\mu = m/L$ - which we can do since the rope is uniform - the weight of the rope below point y is $y\mu$. The tension in the rope at that point is then just $y\mu g$, and the wave speed is

$$v(y) = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{y\mu g}{\mu}\mu} = \sqrt{yg}$$

(b) The time a wave takes to travel the distance of the rope is found by noting that $v(y) = dy/dt$ and integrating along the length of the rope:

$$v(y) = \frac{dy}{dt}$$

$$dt = \frac{dy}{v(y)}$$

$$\Delta t = \int_0^L \frac{dy}{v(y)} = \int_0^L \sqrt{\frac{1}{yg}}\,dy = 2\sqrt{\frac{y}{g}}\Bigg|_0^L = 2\sqrt{\frac{L}{g}}$$

12. Halliday, Resnick & Walker Problem 16.34. A sinusoidal wave of angular frequency $\omega = 1200\,\mathrm{rad/s}$ and amplitude $3.00\,\mathrm{mm}$ is sent along a cord with linear density $2.00\,\mathrm{g/m}$ and tension $1200\,\mathrm{N}$.

(a) The average rate of energy transfer can be found from the angular frequency ω, amplitude y_m, linear density μ, and tension T if we note that the wave speed is $v = \sqrt{T/\mu}$:

$$\mathscr{P}_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2 = \frac{1}{2}\mu\left(\sqrt{\frac{T}{\mu}}\right)\omega^2 y_m^2 = \frac{1}{2}\omega^2 y_m^2 \sqrt{T\mu} \approx 10\,\mathrm{W}$$

(b) Two strings, twice as much power ... the waves cannot interfere if they travel on separate strings 20 W.

(c) Now the waves are along the same string, and we must consider interference. If the two waves have a phase difference φ, they sum to form a new wave whose *amplitude* is given by $2y_m \cos\frac{\varphi}{2}$. The power transmitted by this resultant wave is found by replacing the amplitude of a single wave y_m in our power equation above with $2y_m \cos\frac{\varphi}{2}$:

$$\mathscr{P}'_{\text{avg}} = \frac{1}{2}\omega^2 y_m^2 \sqrt{T\mu}\left[4\cos^2\frac{\varphi}{2}\right] = \left[4\cos^2\frac{\varphi}{2}\right]\mathscr{P}_{\text{avg}}$$

In the first case, we have $\varphi=0$, and since $4\cos^2\frac{\varphi}{2}=4$ we have four times our previous power, $\mathscr{P}'_{\text{avg}} \approx 40\,\text{W}$.

(d) With $\varphi=0.4\pi$, $4\cos^2\frac{\varphi}{2} \approx 2.62$, and thus $\mathscr{P}'_{\text{avg}} \approx 26.2\,\text{W}$.

(e) With $\varphi=\pi$, we have perfect destructive interference: $\cos^2\frac{\varphi}{2}=0$ and thus $\mathscr{P}'_{\text{avg}}=0$.

13. Halliday, Resnick & Walker Problem 16.60. A string tied to a sinusoidal oscillator . . .

The modes of a standing wave on a string of length L are determined only by the mode number n, the length of the string L, and the wave speed v. The wave speed can be determined from the tension in the string T and its linear density μ:

$$f = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

What we have are two different modes n and m which correspond to different tensions T_1 and T_2. In each case, the tension is provided by the hanging mass, so $T_1 = m_1 g$ and $T_2 = m_2 g$. We do not have enough information to so proceed based solely on the equation above and either single mass.

We are told that masses m_1 and m_2 result in standing wave patterns, but *no masses in between* m_1 *and* m_2. This can only be the case if the two masses result in standing wave patterns one mode apart, which would mean that there are no stable wave patterns for masses in between m_1 and m_2. The mode indices must then be adjacent integers. Let the lower mode be n, and the higher $n+1$. From the equation above, the higher mode $n+1$ must correspond to the smaller mass m_1 if f is to be constant.

The frequency of oscillation f is fixed by the oscillator when either mass is present, and thus

$$\begin{aligned}
f = \frac{n}{2L}\sqrt{\frac{m_2 g}{\mu}} &= \frac{n+1}{2L}\sqrt{\frac{m_1 g}{\mu}} \\
n\sqrt{m_2} &= (n+1)\sqrt{m_1} \\
n\left(\sqrt{m_2}-\sqrt{m_1}\right) &= \sqrt{m_1} \\
n &= \frac{\sqrt{m_1}}{\sqrt{m_2}-\sqrt{m_1}} = 4
\end{aligned}$$

The standing wave modes of the string must be the fourth and fifth modes. Now we may rearrange our first equation to solve for μ. Plugging in $n=4$ and $m_2=0.447\,\text{kg}$,

$$\mu = \frac{n^2 m_2 g}{4f^2L^2} \approx 8.46\times10^{-4}\,\text{kg/m}$$

Fluids / Kinetic Theory

14. Two objects, A and B, are submersed in a liquid of density ρ_s at depths of h_A and h_B, respectively. The pressure above the liquid's surface is P_0. What is the difference in pressure experienced by the two objects?

- ▫ $\rho_s g(h_A - h_B) + \frac{1}{2}P_0$
- ▫ $\rho_s g(h_A - h_B) + 2P_0$
- ▫ $\rho_s g(h_A - h_B) \leftarrow$
- ▫ $P_0 + \rho_s g(h_A - h_B)$

15. Viscosity of most liquids can be represented by an extra "drag" force on a body moving in a liquid, which is reasonably well approximated by $F_{\text{drag}} \propto \eta v$, where v is the velocity of the body and η is a parameter of the fluid (in full form, $F_{\text{drag}} = 6\pi\eta Rv$). The presence of viscosity leads to a "terminal velocity" of a body falling in a liquid (*e.g.,* a person falling in air).

Consider a sphere of radius R and density ρ_s falling through a liquid of density ρ and viscosity parameter η. Including this new drag force, the buoyant force, and the weight of the object, which of the following *could* be an expression for the terminal velocity of the sphere?

- ▫ $v = \frac{4R^2(\rho_s - \rho)}{9\eta}$
- ▫ $v = 2R^2\left(\rho_s - \rho\right)$
- ▫ $v = \frac{2R^2\rho_s g}{9\eta}$
- ▫ $v = \frac{2R^2(\rho_s - \rho)g}{9\eta} \leftarrow$

16. Two cylinders A and B have the same volume and contain the same number of moles of a monatomic ideal gas. It is found that the pressure in vessel A is twice the pressure in vessel B. What is the relation between the temperatures of the vessels?

- ▫ $T_A = 2T_B \leftarrow$
- ▫ $T_A = T_B$
- ▫ $T_A = 0.5T_B$
- ▫ $T_A = 4T_B$

17. Estimate the pressure exerted on your eardrum due to the water above when you are swimming at the bottom of a pool that is 5.0 m deep. (Note $\rho_{\text{water}} = 1000\,\text{kg/m}^3$).

- – 4.9×10^4 Pa
- ▫ 1.88×10^5 Pa
- ▫ 2.73×10^6 Pa
- ▫ 3.76×10^5 Pa

The air inside the middle ear is going to stay at about atmospheric pressure. The net force is therefore only the increase of pressure with depth, we do not need to add in the atmospheric pressure above the water.

$$\Delta P = \rho g h = [1000\,\mathrm{kg/m}]\left[9.81\,\mathrm{m/s^2}\right][5\,\mathrm{m}] = 4.9\times10^4\mathrm{Pa}$$

18. An automobile tire is inflated with air originally at 10.0°C and normal atmospheric pressure. During the process, the air is compressed to 26.0% of its original volume and the temperature is increased to 32.0°C. What is the tire pressure?

- □ 7.15×10^4 Pa
- □ 8.35×10^5 Pa
- – 4.20×10^5 Pa
- □ 1.23×10^6 Pa

Use the ideal gas law, $PV = nRT$ to start with. The constant here is the number of moles, $n = PV/RT$, so we relate PV/RT before and after the inflation, remembering to keep temperature in Kelvins ($T_K = 273.15 + T_C$)

$$\frac{P_1V_1}{RT_1} = \frac{P_2V_2}{RT_2}$$

$$\frac{P_{\mathrm{atm}}V_1}{283} = \frac{P_2\cdot 0.26V_1}{305}$$

$$P_2 = \left[\frac{305}{283}\right]\left[\frac{1}{0.26}\right]P_{\mathrm{atm}}$$

$$P_2 = 4.20\times10^5\ \mathrm{Pa}$$

Thermal Physics

19. 1 kg of liquid nitrogen at its boiling point of −195.81°C is in an isolated container of negligible mass. A mass of liquid water m_w at 25°C is dropped into the container. What should m_w be in order to boil away (vaporize) all of the liquid nitrogen, leaving behind ice at 0°C? You may need the following data:

liquid nitrogen	$L_v = 2.01\times10^5$ J/kg
water	$c = 4190$ J/kg·K $L_f = 3.34\times10^5$ J/kg $T = 0°C$ freezing point

20. A pot with a steel bottom 8.50 mm thick rests on a hot stove. The area of the bottom of the pot is 0.150 m^2. The water inside the pot is at 100.0°C, and 0.390 kg of water are evaporated every 3.00 min. Find the temperature of the lower surface of the pot, which is in contact with the stove. The thermal conductivity of steel is $k = 50.2$ W/m K, and the latent heat of vaporization of water is $L_v = 2.256\times10^6$ J/kg.

21. How much heat must be absorbed by ice of mass $m = 0.72$ kg at $T_i = -20°\mathrm{C}$ to bring it to a liquid state at $T_f = 15°\mathrm{C}$?

- ■ 317 kJ
- □ 187 kJ
- □ 207 kJ

- ▫ 97 kJ

22. In the previous question, which step in the melting and heating process requires the *greatest* heat input?

- ▫ warming the ice
- ■ melting the ice
- ▫ warming the liquid

23. In the previous question, which step in the melting and heating process requires the *smallest* heat input?

- ■ warming the ice
- ▫ melting the ice
- ▫ warming the liquid

24. A 0.050 kg ingot of metal is heated to 200°C and dropped into a beaker containing 0.400 kg of water initially at 20.0°C. If the final equilibrium temperature is 22.4°C, what is the specific heat c of the metal? Ignore heat transferred to the beaker and boil-off of the water. Assume the system is isolated. (Note: $c_{\text{water}} = 4186\,\text{J/kg}\cdot\,\text{K}$.)

- ▫ 279 J/kg·°C
- ▫ 148 J/kg·°C
- ▫ 721 J/kg·°C
- ■ 453 J/kg·°C

25. The temperature of a silver bar rises by 10°C when it absorbs 1.23 kJ of energy by heat. The mass of the bar is 525 g. Determine the specific heat c of silver.

- ■ 234 J/kg· K
- ▫ 1240 J/kg· K
- ▫ 1.95 J/kg· K
- ▫ 8820 J/kg· K