University of Alabama
Department of Physics and Astronomy

PH 105 LeClair — Summer 2015

Week 1 Homework - Solution

Some of the problems here are from "Problems and Solutions in Introductory Mechanics" by David Morin, a nice (and very inexpensive) book you might find useful as a supplement.[i] It has a nice but very brief summary of every chapter, but most of the book is just problems and carefully-worked solutions. If you feel like you're not seeing enough example problems, this will solve your problem for about $15. I've noted which problems I've used are to give proper attribution. Similarly, other problems are from your textbook, most those are noted as well.

Problems for 26 May (due 27 May)

1. *HRW 1.30* Water is poured into a container that has a leak. The mass m of the water is as a function of time t is

$$m = 5.00t^{0.8} - 3.00t + 20.00$$

with $t \geqslant 0$, m in grams, and t in seconds. At what time is the water mass greatest?

Solution: We'll follow the template approach on this one to give you an idea of how it is supposed to work.

Given: Water mass versus time $m(t)$.

Find: The time t at which the water mass m is greatest. This can be accomplished by finding the time derivative of $m(t)$ and setting it equal to zero, followed by checking the second derivative to be sure we have found a maximum.

Sketch: It is useful to plot the function $m(t)$ and graphically estimate about where the maximum should be, roughly.[ii] From the plot on the next page, it is clear that there is indeed a maximum water mass, and it occurs just after $t = 4\,\text{s}$.

Relevant equations: We need to find the maximum of $m(t)$. Therefore, we need to set the first derivative equal to zero. We must also check that the second derivative is negative to ensure that we have found a maximum, not a minimum. Therefore, only two equations are needed:

[i] See http://www.people.fas.harvard.edu/~djmorin/book.html or look it up on Amazon.

[ii] It is relatively easy to do this on a graphing calculator, which can be found online these days: http://www.coolmath.com/graphit/.

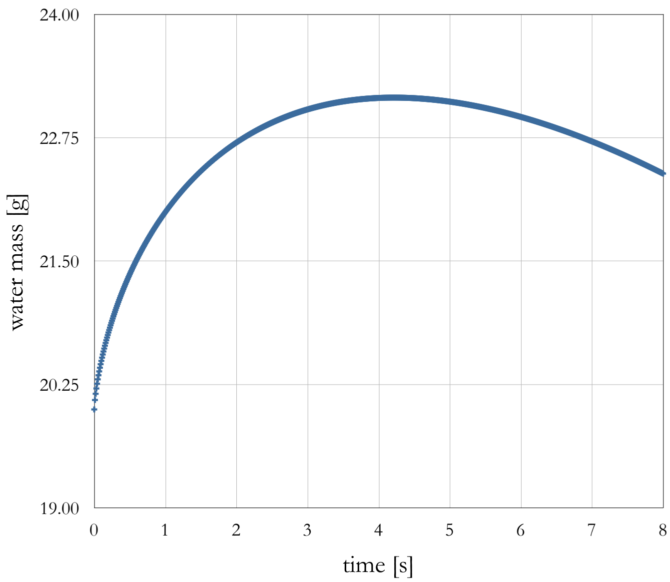


Figure 1: *Water mass versus time, problem 1. Note the rather expanded vertical axis, with offset origin.*

$$\frac{dm}{dt} = \frac{d}{dt}\left[m(t)\right] = 0 \qquad \text{and} \qquad \frac{d^2m}{dt^2} = \frac{d^2}{dt^2}\left[m(t)\right] < 0 \quad \Longrightarrow \quad \text{maximum in } m(t)$$

Symbolic solution:

$$\begin{aligned}
\frac{dm}{dt} &= \frac{d}{dt}\left[5t^{0.8} - 3t + 20\right] = 0.8\left(5t^{0.8-1}\right) - 3 = 4t^{-0.2} - 3 = 0 \\
4t^{-0.2} - 3 &= 0 \\
t^{-0.2} &= \frac{3}{4} \\
\Longrightarrow \quad t &= \left(\frac{3}{4}\right)^{-5} = \left(\frac{4}{3}\right)^{5}
\end{aligned}$$

Thus, $m(t)$ takes on an extreme value at $t = (4/3)^5$. We did not prove whether it is a maximum or a minimum however! This is important . . . so we should apply the *second derivative* test.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway:

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left[\frac{dm}{dt}\right] = \frac{d}{dt}\left[4t^{-0.2} - 3\right] = -0.2\left(4t^{-0.2-1}\right) = -0.8t^{-1.2} \\
\frac{d^2m}{dt^2} &< 0 \quad \forall \quad t > 0
\end{aligned}$$

Since $t^{-1.2}$ is always positive for $t>0$, $\frac{d^2m}{dt^2}$ is always less than zero[iii], which means we have indeed found a maximum.

Numeric solution: Evaluating our answer numerically, remembering that t has units of seconds (s):

$$t = \left(\frac{4}{3}\right)^5 \approx 4.21399 \xrightarrow[\text{digits}]{\text{sign.}} 4.21\,\text{s}$$

The problem as stated has only three significant digits, so we round the final answer appropriately.

Double check: From the plot above, we can already graphically estimate that the maximum is somewhere around $4\frac{1}{4}$ s, which is consistent with our numerical solution to 2 significant figures. The dimensions of our answer are given in the problem, so we know that t is in seconds. Since we solved $dm/dt(t)$ for t, the units must be the same as those given, with t still in seconds – our units are correct.

2. **(a)** Find the separation vector $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ between the vectors $\vec{\mathbf{r}}' = (3, 4, 5)$ and $\vec{\mathbf{r}} = (7, 2, 17)$. **(b)** Determine its magnitude, and **(c)** construct the corresponding unit vector.

Solution: The separation vector is found by subtracting the two vectors. Constructing a unit vector is accomplished by dividing the given vector by its magnitude, making a vector of unit length.

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}' = (7-3)\;\hat{\mathbf{x}} + (2-4)\;\hat{\mathbf{y}} + (17-5)\;\hat{\mathbf{z}} = 4\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + 12\,\hat{\mathbf{z}}$$

$$|\Delta\vec{\mathbf{r}}| = \sqrt{4^2 + (-2)^2 + 12^2} = 2\sqrt{41} \approx 12.8$$

$$\hat{\Delta\vec{\mathbf{r}}} = \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} = \frac{4\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + 12\,\hat{\mathbf{z}}}{2\sqrt{41}} = \frac{1}{\sqrt{41}}\,(2\,\hat{\mathbf{x}} - 1\,\hat{\mathbf{y}} + 6\,\hat{\mathbf{z}})$$

3. *HRW 3.28* Here are two vectors:

$$\vec{\mathbf{a}} = 1.0\,\hat{\imath} + 2.0\,\hat{\jmath} \qquad \vec{\mathbf{b}} = 3.0\,\hat{\imath} + 4.0\,\hat{\jmath}$$

Find the following quantities:

a) the magnitude of $\vec{\mathbf{a}}$

b) the angle of $\vec{\mathbf{a}}$ relative to $\vec{\mathbf{b}}$

c) the magnitude and angle of $\vec{\mathbf{a}} + \vec{\mathbf{b}}$

[iii] You can read the symbol $\forall$ above as "for all." Thus, $\forall\ t>0$ is read as "for all t greater than zero."

d) the magnitude and angle of $\vec{\mathbf{a}} - \vec{\mathbf{b}}$

Solution: **(a)** If $\vec{\mathbf{a}}$ is in general a vector defined by $\vec{\mathbf{a}} = a_x \hat{\imath} + a_y \hat{\jmath}$, the magnitude of $\vec{\mathbf{a}}$ is found by

$$|\vec{\mathbf{a}}| = \sqrt{a_x^2 + a_y^2} \tag{1}$$

In the present case, this gives $|a| = \sqrt{5}$.

(b) The angle θ between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is most easily found using the scalar product:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos\theta = a_x b_x + a_y b_y \tag{2}$$

Thus,

$$\cos\theta = \frac{a_x b_x + a_y b_y}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \frac{3+8}{\left(\sqrt{5}\right)(5)} = \frac{11}{5\sqrt{5}} \tag{3}$$

$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right) \approx 10.3° \tag{4}$$

(c) First we'll need the vector sum

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_x + b_x)\,\hat{\imath} + (a_y + b_y)\,\hat{\jmath} = 4\hat{\imath} + 6\hat{\jmath} \tag{5}$$

The magnitude is easy enough

$$|\vec{\mathbf{a}} + \vec{\mathbf{b}}| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13} \tag{6}$$

The tangent of the angle with the horizontal axis is the ratio of the y and x components:

$$\tan\theta = \frac{a_y + b_y}{a_x + b_x} = \frac{6}{3} = 2 \tag{7}$$

$$\theta = \tan^{-1} 2 = 63.4° \tag{8}$$

(d) Same deal, but we reverse the sign of $\vec{\mathbf{b}}$ to perform subtraction.

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}}) = (a_x - b_x)\,\hat{\imath} + (a_y - b_y)\,\hat{\jmath} = -2\hat{\imath} + -2\hat{\jmath} \tag{9}$$

The magnitude proceeds as it did in the last part ...

$$|\vec{a} - \vec{b}| = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2} \tag{10}$$

... as does the determination of the angle:

$$\tan\theta = \frac{a_y - b_y}{a_x - b_x} = \frac{-2}{-2} = 1 \tag{11}$$

$$\theta = \tan^{-1} 1 = 45^\circ \tag{12}$$

The angle the vector makes with respect to the horizontal axis is 45°. We should be careful, however: knowing that both components are negative, we know the vector points down and to the left. This tells us that the vector is pointing 45° *below* the horizontal axis, and backward along the $-y$ direction. If we want to be more precise, we might say the vector makes an angle of 225° with the x axis.

Problems for 27 May (due 28 May)

4. *Morin MC 2.3* If the acceleration as a function of time is known to be $a(t) = \alpha t$, and if $x = v = 0$ at $t = 0$, what is the position versus time $x(t)$?

Solution: We know that $a = dv/dt$, so integrating the acceleration gives us the velocity, which we can integrate to get the position.

$$v(t) = \int a(t)\, dt = \int \alpha t\, dt = \frac{1}{2}\alpha t^2 + C \tag{13}$$

What is the constant C? We have to use the given data, specifically the fact that $v(0) = 0$, to find it.

$$v(0) = \frac{1}{2}\alpha (0)^2 + C = C = 0 \qquad \Longrightarrow \qquad C = 0 \tag{14}$$

Thus, $v(t) = \frac{1}{2}\alpha t^2$. Now since $v = dx/dt$, we can integrate once more and find the position.

$$x(t) = \int v(t)\, dt = \int \frac{1}{2}\alpha t^2\, dt = \frac{1}{2}\alpha \left(\frac{t^3}{3}\right) + C' \tag{15}$$

Again, we find the constant C' by using the fact that $x(0) = 0$, which gives $C' = 0$. Thus, the position versus time is

$$x(t) = \frac{1}{6}\alpha t^3 \tag{16}$$

5. *Morin MC 2.9* You are driving a car that has a maximum acceleration of a. The magnitude of the maximum deceleration is also a. What is the maximum distance you can travel in a time T, assuming you begin and end at rest?

Solution: Since the magnitudes of acceleration and deceleration are the same, the maximum position will happen when you spend half your time speeding up and half your time slowing down. This means that during the speeding up (accelerating) phase you'll cover half the total distance, and you'll cover the other half while slowing down (decelerating). The question is really then given acceleration a and a time $T/2$, how far does one go? Double that number is the total distance covered.

Presume the car starts from the origin at rest. We know then that $x(0)=v(0)=0$, so the equation of motion for the car is

$$x(t) = \frac{1}{2}at^2 \tag{17}$$

For the accelerating phase, a is positive, for the decelerating phase, a is negative. Doesn't really matter though, since we only need to worry about one or the other and double the result, since they cover the same distance. Either way, after a time $T/2$ we cover a distance

$$\frac{1}{2}a\left(\frac{T}{2}\right)^2 = \frac{1}{8}aT^2 \tag{18}$$

That's how far you would travel during the acceleration phase, and since you'd cover the same distance during the deceleration phase the total is

$$x_{\text{total}} = 2x\left(\frac{T}{2}\right) = \frac{1}{4}aT^2 \tag{19}$$

6. *Morin MC 2.7* An object starts from rest at the origin at time $t=-T$ and accelerates with constant acceleration a. A second object starts from rest at the origin at time $t=0$ and accelerates with the same a. How far apart are they at time t? Explain the meaning of the two terms in your answer.

Solution: The main trick here is to keep track of the time in our equations very carefully. Object 1 starts moving T seconds earlier than object 2. If our clock starts at $t=0$ when object 2 starts moving, that means object 1 has been moving for T seconds already. Therefore, wherever we would normally write t in our equations for object 1, we should use $(t+T)$ to account for the head start.

For object 1, we know that $v_1(0) = x_1(0) = 0$ since the object starts at the origin at rest. That means its equation of motion, accounting for the head start, is

$$x_1(t) = \frac{1}{2}a\,(t+T)^2 \tag{20}$$

For object 2, it is also true that $v_2(0)=x_2(0)=0$. Since there is no head start for object 2, we just have

$$x_2(t) = \frac{1}{2}at^2 \tag{21}$$

The difference between the two is

$$x_1(t) - x_2(t) = \frac{1}{2}a\,(t+T)^2 - \frac{1}{2}at^2 = \frac{1}{2}at^2 + aTt + \frac{1}{2}aT^2 - \frac{1}{2}at^2 = aTt + \frac{1}{2}aT^2 \tag{22}$$

The two terms here have a simple physical interpretation. In short, the first object has a head start of $\frac{1}{2}aT^2$ and pulls away from the second at a rate of $(aT)t$. This is worth explaining a bit further. The second term, $\frac{1}{2}T^2$, is the distance the first object traveled before the second one started, its head start. The first term represents the fact that since the first object has been accelerating longer, it will always be going faster than the second, and the two will always have a relative speed between them. Their difference in speed would just be the speed the first object had at $t=0$ when the second object gets started, or aT. This constant *difference* in speed translates to a linearly increasing *difference* in position of $(aT)t$ in the same way any constant speed leads to a linearly increasing position.

7. *Morin MC 2.9* A ball is dropped from rest at height h. Directly below on the ground, a second ball is simultaneously thrown upward with speed v_0. **(a)** If the two balls collide at the moment the second ball is instantaneously at rest, what is the height of the collision? **(b)** What is the relative speed of the balls when they collide? Ignore air resistance.

Solution: Let the dropped ball have position $x_1(t)$ and the thrown ball $x_2(t)$, with $t=0$ when the two balls are released. For simplicity, let $+x$ be upward, with the origin at ground level. This gives both balls an acceleration of $-g$. Now we can readily write down their positions at any time, given the starting height h of the first ball and the initial velocity v_0 of the second:

$$x_1(t) = h - \frac{1}{2}gt^2 \tag{23}$$

$$x_2(t) = v_0t - \frac{1}{2}gt^2 \tag{24}$$

First we can find the time when the second ball is at rest

$$v_2(t) = \frac{dx_2}{dt} = v_0 - gt = 0 \tag{25}$$

$$\implies \quad t = \frac{v_0}{g} \tag{26}$$

At this time, the position of both balls should be the same if they are to collide.

$$x_1(\frac{v_0}{g}) = h - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = h - \frac{v_0^2}{2g} \tag{27}$$

$$x_2(\frac{v_0}{g}) = v_o\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} \tag{28}$$

$$\implies \quad h - \frac{v_0^2}{2g} = \frac{v_0^2}{2g} \tag{29}$$

$$\implies v_0 = \sqrt{gh} \tag{30}$$

This relates the initial velocity of the second ball to the starting height of the first ball. Using this in either $x_1(t)$ or $x_2(t)$ along with the previously found time gives us an expression for the height of the collision:

$$x_2(\frac{v_0}{g}) = \frac{v_0^2}{2g} = \frac{gh}{2g} = \frac{h}{2} \tag{31}$$

The balls collide at exactly half the starting height of the first ball, at a time $t = \frac{v_0}{g} = \sqrt{\frac{h}{g}}$. Their relative speed at the time of the collision is also readily found:

$$v_1(t) - v_2(t) = -gt - v_0 + gt = -v_0 \tag{32}$$

In fact, at *any* time the difference in the balls' speeds is v_0 - this is the relative speed they start out with, and since we have only the influence of gravity to worry about, their speeds at any later time changes by exactly the same amount, $-gt$.

8. *Morin MC 2.12* You throw a ball upward. After half of the time to the highest point, the ball has covered what fraction of its maximum height? Ignore air resistance.

Solution: Another problem from David Morin's book. Let the ground level be $x = 0$, with $+x$ running upward. The ball's position at any time, assuming an initial velocity v_0, is then[iv]

[iv]We are not given the initial velocity, but we need it to work the problem. In most cases like this, quantities you had to introduce yourself will be part of the calculation and not the final answer. In this case, we introduced v_0 to solve the problem, but since it wasn't specified our final answer should be independent of v_0.

$$x(t) = v_0 t - \frac{1}{2} g t^2 \tag{33}$$

The time to its highest point is found by maximizing $x(t)$, or equivalently, finding the time at which the velocity is zero.

$$v(t) = \frac{dx}{dt} = v_0 - gt = 0 \tag{34}$$

$$\implies \quad t_{max} = \frac{v_0}{g} \tag{35}$$

At this time, we can find the ball's height:

$$x\left(t_{max}\right) = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} \tag{36}$$

At half this time, the ball's height is

$$x\left(\frac{1}{2}t_{max}\right) = v_0\left(\frac{v_0}{2g}\right) - \frac{1}{2}g\left(\frac{v_0}{2g}\right)^2 = \frac{3v_0^2}{8g} \tag{37}$$

The fraction of maximum height is then

$$\text{fraction of max height} = \frac{x(\frac{1}{2}t_{max})}{x(t_{max})} = \frac{\frac{3v_0^2}{8g}}{\frac{v_0^2}{2g}} = \frac{3}{4} \tag{38}$$

Since the ball is going much faster during the first half of its motion, it covers more distance. The last half of the ball's flight only covers 1/4 of the net vertical distance.

Problems for 28 May (due 29 May)

9. *Morin MC 3.7* Two balls are thrown with the same speed v_o from the top of a cliff. The angles of their initial velocities are θ above and below the horizontal (i.e., one is thrown downward at θ below the horizontal, one is thrown upward at the same angle). How much farther along the ground does the top ball hit compared to the bottom ball? *Hint: The two trajectories have a part in common. Not much calculation is necessary.*

Solution: Think first about the ball launched above the horizontal. If it leaves the cliff at an angle θ with speed v_o, when it comes back to the same height, it have the same speed v_o but will be directed *downward* with angle $-\theta$. This will always be true for projectiles without air resistance - same height give the same velocity, and the magnitude of the angle relative to the horizontal will also be the same (though its sign does change).

What that means is that once the ball launched upward comes back to its original height, it will have the same velocity as the ball launched downward, and the two will cover exactly the same distance from that point onward. Therefore we really just need to figure out how far the upward-launched ball goes before it comes back to its original height, which is just the range of the projectile, $v_o^2 \sin 2\theta/g$.

What if you didn't know that trick? If the height of the cliff is h, and we place the origin at the launch point, we can write down the $y(x)$ equation for both projectiles.

$$y_1(x) = x_1 \tan\theta - \frac{gx_1^2}{2v_o^2\cos^2\theta} \tag{39}$$

$$y_2(x) = -x_2 \tan\theta - \frac{gx_2^2}{2v_o^2\cos^2\theta} \tag{40}$$

Both will hit the ground when $y\!=\!-h$, since the ground is a distance h below the launch point.

$$-h = x_1 \tan\theta - \frac{gx_1^2}{2v_o^2\cos^2\theta} \tag{41}$$

$$-h = -x_2 \tan\theta - \frac{gx_2^2}{2v_o^2\cos^2\theta} \tag{42}$$

Solve the first one for x_1, the second for x_2 (taking the positive roots in both cases), and $x_1\!-\!x_2$ is how much farther the upward-launched ball goes. It is tedious, but it does work.

10. *Morin PR 3.5*A ball is thrown with speed v at angle θ with respect to horizontal ground. At the highest point in the motion, the strength of gravity is somehow magically doubled. What is the total horizontal distance traveled by the ball?

Solution: During the first half of the motion, gravity is acting normally, and we just need to figure out where the projectiles is along x at the moment the y velocity is zero.

$$v_y(t) = v_o \sin\theta - gt = 0 \qquad \Longrightarrow \qquad t_{\text{max}} = \frac{v_o \sin\theta}{g} \tag{43}$$

$$x_1(t) = (v_o \cos\theta)\, t = v_o \cos\theta \left(\frac{v_o \sin\theta}{g}\right) = \frac{v_o^2 \sin\theta\cos\theta}{g} \tag{44}$$

For the second half of the motion, where gravity is twice as strong, we now first need to figure out how long the ball will be in the air. If we know the maximum height, this is easy enough. From our result above, we can use the time we found for the ball to be at maximum and substitute that into the $y(t)$ equation.

$$y_1(t) = (v_o \sin\theta)\, t - \frac{1}{2} g t^2 \tag{45}$$

$$y_{max} = y_1(t_{max}) = v_o \sin\theta \left(\frac{v_o \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_o \sin\theta}{g}\right)^2 = \frac{v_o^2 \sin^2\theta}{2g} \tag{46}$$

Falling from a height y_{max} with an acceleration $2g$, the equation of motion for the second half of the flight would be

$$y_2(t) = y_{max} - \frac{1}{2}(2g)\, t^2 = \frac{v_o^2 \sin^2\theta}{2g} - \frac{1}{2}(2g)\, t^2 \tag{47}$$

We can find out how long this takes by setting $y_2(t) = 0$, and using that time we can find the additional horizontal distance

$$y_2(t) = \frac{v_o^2 \sin^2\theta}{2g} - \frac{1}{2}(2g)\, t^2 = 0 \qquad \Longrightarrow \qquad t_2 = \frac{v_o \sin\theta}{g\sqrt{2}} \tag{48}$$

$$x_2(t_2) = (v_o \cos\theta)\, t = (v_o \cos\theta)\, \frac{v_o \sin\theta}{g\sqrt{2}} = \frac{v_o^2 \sin\theta \cos\theta}{g\sqrt{2}} \tag{49}$$

The total horizontal distance covered is then

$$x_{tot} = x_1 + x_2 = \frac{v_o^2 \sin\theta\cos\theta}{g}\left(1 + \frac{1}{\sqrt{2}}\right) \tag{50}$$

11. *Morin PR 3.10* **(a)** You wish to throw a ball to a friend who is a distance $2d$ away, and you want the ball to just barely clear a wall of height h that is located halfway to your friend. At what angle θ should you throw the ball? **(b)** What initial speed v_o is required? What value of h (in terms of d) yields the minimum v_o? What is the value of θ in this minimum case?

Solution: The throw requires a projectile range of $2d$, and a maximum height of h since the wall is at the halfway point.

$$R = 2d = \frac{v_o^2 \sin 2\theta}{g} = \frac{2v_o^2 \sin\theta\cos\theta}{g} \tag{51}$$

$$h = \frac{v_o^2 \sin^2\theta}{2g} \tag{52}$$

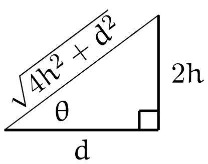
Now we have two equations, and two unknowns: θ and v_o. We can solve one equation for v_o and plug it in the other, or in this case, just divide the two equations to eliminate v_o. This results in

$$\frac{h}{2d} = \frac{v_o^2 \sin^2\theta}{2g} \frac{g}{2v_o^2 \sin\theta\cos\theta} \tag{53}$$

$$\frac{h}{d} = \frac{\sin\theta}{2\cos\theta} = \frac{1}{2}\tan\theta \tag{54}$$

$$\tan\theta = \frac{2h}{d} \tag{55}$$

That gives us the angle, given h and d. How about the velocity? We could use either the height or range equation, given θ, with the height equation being marginally easier. The height equation involves $\sin\theta$, so how do we find that given that we know the tangent? Given $\tan\theta = \frac{2h}{d}$, the right triangle below is implied.



$$\tan\theta = \tfrac{2h}{d}$$
$$\sin\theta = \tfrac{2h}{\sqrt{d^2+4h^2}}$$

Now we can use our height equation and solve it for v_o

$$h = \frac{v_o^2 \sin^2\theta}{2g} \tag{56}$$

$$v_o^2 = \frac{2gh}{\sin^2\theta} = \frac{g\left(d^2+4h^2\right)}{2h} \tag{57}$$

$$v_o^2 = \sqrt{\frac{g\left(d^2+4h^2\right)}{2h}} \tag{58}$$

What angle gives the minimum v_o required? The one that has $dv_o/dh = 0$. It is somewhat tedious to find the derivative, but Wolfram Alpha is quite good at it. You should find a minimum velocity for a height of $h = d/2$, giving an angle of exactly 45° - the same as the angle for maximum range.

12. *Morin PR 3.11* A person throws a ball with speed v_o at a 45° angle and hits a given target. How much quicker does the ball get to the target if the person instead throws the ball with the same speed, but at an angle that makes the trajectory consist of two identical bumps? (Assume unrealistically that there is no loss in speed at the bounce.)

Solution: The range at 45° is just $R_1 = v_o^2/g$. Throwing at a lower angle, we want to have exactly half the range, which implies

$$R_2 = \frac{v_o^2 \sin 2\theta}{g} = \frac{R_1}{2} = \frac{v_o^2}{2g} \tag{59}$$

Inspection of the above implies that $\sin 2\theta = \frac{1}{2}$, or that $\theta = 15°$. What is the time in each case? In the first case, it is just the distance R divided by the horizontal velocity.

$$\text{horizontal distance} = R_1 = v_x t_1 = v_o \cos 45° t_1 \tag{60}$$

$$t_1 = \frac{R_1}{v_o \cos 45°} = \frac{\sqrt{2}R_1}{v_o} \tag{61}$$

In the second case, each bounce covers a horizontal distance $R_2 = R_1/2$, and does so at a velocity $v_x = v_o \cos 15°$. We have to double the time we find this way to account for both bounces.

$$\text{horizontal distance, one bounce} = R_2 = \frac{R_1}{2} = v_x t_2 = v_o \cos 15° t_2 \tag{62}$$

$$t_2 = \frac{R_1}{2v_o \cos 15°} \tag{63}$$

$$t_{2,\,\text{tot}} = 2t_2 = \frac{R_1}{v_o \cos 15°} = \frac{2\sqrt{2}R_1}{v_o\left(1+\sqrt{3}\right)} \tag{64}$$

For the last step, one has to know that $\cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}} \approx 0.966$, but using the exact result is not necessary. So how much faster is it to bounce the ball? Let's see.

$$\Delta t = t_1 - t_{2,\,\text{tot}} = \frac{\sqrt{2}R_1}{v_o} - \frac{2\sqrt{2}R_1}{v_o\left(1+\sqrt{3}\right)} = \frac{\sqrt{2}R_1}{v_o}\left(1 - \frac{2}{1+\sqrt{3}}\right) = t_1\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \approx 0.27t_1 \tag{65}$$

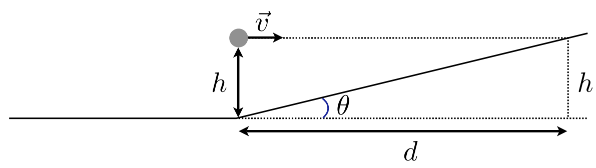
Bouncing is about 27% faster than the direct throw. Next time you watch a baseball game, keep an eye out: this happens all the time.

13. *HRW 2.80* A pilot flies horizontally at 1300 km/h, at height $h = 35\,\text{m}$ above initially level ground. However, at time $t = 0$, the pilot begins to fly over ground sloping upward at angle $\theta = 4.3°$. If the pilot does not change the airplane's heading, at what time t does the plane strike the ground?

Solution: **Given:** The initial velocity and height of a plane flying toward an upward slope of angle θ.

Find: How long before the plane hits the slope? At time $t = 0$, the plane is at the beginning of the slope, a height h above level ground. Assuming the plane continues at the same horizontal speed, we wish to find the time at which the plane hits the slope. Given the plane's velocity and height and the slope's angle, we can relate the horizontal distance to intercept the ramp to the plane's height.

Sketch: Assume a spherical plane (it doesn't matter). If the plane is at altitude h, it will hit the ramp after covering a horizontal distance d, where $\tan\theta = h/d$.



Relevant equations: We can relate the horizontal distance to intersect the ramp to the plane's altitude using the known slope of ground:

$$\tan\theta = \frac{h}{d}$$

We can determine how long the horizontal distance d will be covered given the plane's constant horizontal speed v:

$$d = vt$$

Symbolic solution: Combining our equations above, the time t it takes for the plane to hit the slope is

$$t = \frac{d}{v} = \frac{h}{v\tan\theta}$$

Numeric solution: Using the numbers given, and converting units,

$$t = \frac{h}{v\tan\theta} = \frac{35\,\text{m}}{1300\,\text{km/h}\,(1000\,\text{m/km})\,(1\,\text{h}/3600\,\text{s})\,(\tan 4.3°)} \approx 1.3\,\text{s}$$

Problems for 29 May (due 1 June)

14. *HRW 4.50* Two seconds after being projected from ground level, a projectile is displaced 40 m horizontally and 53 m vertically above its launch point. What are the horizontal and vertical components of the initial velocity of the projectile?

Solution: *HRW 4.50* We know that at time t a projectile is at position (x, y). For convenience, we will define the launch position to be the origin of our coordinate system. Presuming the y axis to be vertical, with gravitational acceleration along $-y$, we can describe the position of the projectile at any given time t:

$$y(t) = v_{iy}t - \frac{1}{2}gt^2 \tag{66}$$
$$x(t) = v_{ix}t \tag{67}$$

Here v_{ix} and v_{iy} are respectively the x and y components of the initial (launch) velocity. Given that we know $x(t)$ and $y(t)$ at the time of interest, all we need to do is solve the equations above for velocity instead of position.

$$v_{iy} = \frac{y + \frac{1}{2}gt^2}{t} \tag{68}$$
$$v_{ix} = \frac{x}{t} \tag{69}$$

Given the position of the particle is $(x, y) = (40, 53)$ at time $t = 2.0\,\mathrm{s}$, a numerical solution is easy:

$$v_{iy} = 36\,\mathrm{m/s} \tag{70}$$
$$v_{ix} = 20\,\mathrm{m/s} \tag{71}$$

15. A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\vec{v}_i$ as shown below. What must be its minimum initial speed if the ball is never to hit the rock after it is kicked? Note this is not circular motion.

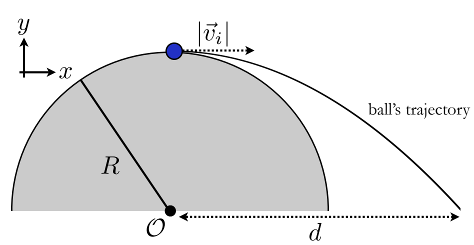


Figure 2: *A ball is kicked off the top of a rock by an unseen person.*

Solution: **Find:** The minimum speed for the ball not to hit the rock. As long as we're at it, we will also find the net horizontal distance it lands from the rock at that speed. Since the rock may be described by a circle, and the ball's motion a parabola, we are seeking a condition on the initial velocity such that the parabola always lies above the circle.

Given: The geometry of the rock, the ball's initial velocity.

Sketch: Let the x axis run horizontally and the y axis vertically, with the origin at the rock's center. This makes the ball's starting position $(0, R)$ and its launch angle with respect to the x axis $\theta = 0$ (as shown in the figure included in the problem).

Relevant equations: We need the equation of a circle of radius R centered on the origin, and the trajectory of a projectile fired at angle $\theta = 0$ relative to the x axis with starting vertical position $y(0) = R$. Let the circle be described by $y_p(x)$ and the rock $y_r(x)$. Since our solution is restricted to the upper right quadrant, the rock may be described by

$$y_r(x) = \sqrt{R^2 - x^2} \tag{72}$$

The ball's trajectory is our well-known result

$$y_p(x) = y(0) + (\tan\theta)\, x - \frac{gx^2}{2|\vec{v}|^2 \cos^2\theta} = R - \frac{gx^2}{2|\vec{v}|^2} \tag{73}$$

Symbolic solution: The condition that the ball does not hit the rock is simply that the parabola and circle above have no intersection point, other than the trivial one at $(0, R)$. That is, the parabola must lie above the circle everywhere except $(0, R)$. Thus,

$$\begin{aligned} y_p(x) &\geqslant y_r(x) \\ R - \frac{gx^2}{2|\vec{v}|^2} &\geqslant \sqrt{R^2 - x^2} \end{aligned} \tag{74}$$

In principle, this is it. Much algebra now ensues. First, simply square both sides and simplify. Since both sides must be positive for all x considered, by the problem's construction, this does not alter the inequality.

$$
\begin{aligned}
\left(R - \frac{gx^2}{2|\vec{v}|^2}\right)^2 &\geqslant \left(\sqrt{R^2 - x^2}\right)^2 \\
R^2 - \frac{gRx^2}{|\vec{v}|^2} + \frac{g^2x^4}{4|\vec{v}|^4} &\geqslant R^2 - x^2 \\
\left(\frac{g^2}{4|\vec{v}|^2}\right)x^4 + \left(1 - \frac{gR}{|\vec{v}|^2}\right)x^2 &\geqslant 0 \\
x^2\left(\frac{g^2}{4|\vec{v}|^2}x^2 + 1 - \frac{gR}{|\vec{v}|^2}\right) &\geqslant 0 \\
x^2\left(\frac{g^2}{4|\vec{v}|^2}x^2 + 1 - \frac{gR}{|\vec{v}|^2}\right) &\geqslant 0 \qquad (x \neq 0) \\
\frac{g^2}{4|\vec{v}|^2}x^2 + 1 - \frac{gR}{|\vec{v}|^2} &\geqslant 0 \\
\frac{g}{4|\vec{v}|^2}x^2 &\geqslant \left(\frac{gR}{|\vec{v}|^2} - 1\right)
\end{aligned}
\tag{75}
$$

We require this inequality to be true for all $x > 0$ for the ball not to hit the rock anywhere in the domain of interest. The only way this can happen is if the right-hand side is negative:

$$
\begin{aligned}
&\frac{gR}{|\vec{v}|^2} - 1 \leqslant 0 \\
\implies \quad &|\vec{v}| \geqslant \sqrt{gR}
\end{aligned}
\tag{76}
$$

Note that this is *not* the same condition you would find by simply requiring the particle's *range* to be larger than R. It is easy to verify that one can make a parabola with horizontal range R in this situation that still intersects the circle . . . try it out!

Where does the projectile land? Clearly, at $y_p = 0$, since that is where the ground is. We simply need to set the ball's y position equal to zero, and solve for the resulting x value using our minimal velocity from Eq. 76. This is where the ball lands.

$$
\begin{aligned}
y_p = 0 &= R - \frac{gx^2}{2|\vec{v}|^2} = R - \frac{gx^2}{2gR} = R - \frac{x^2}{2R} \\
2R^2 &= x^2 \\
\implies \quad x &= R\sqrt{2}
\end{aligned}
\tag{77}
$$

What is more interesting is *how far from the rock* the ball lands. Since the rock extends to $x = R$, we have gone beyond that by a distance

$$
\text{distance from rock} = R\left(\sqrt{2} - 1\right) \tag{78}
$$

Numeric solution: Numbers? How awkward. $\sqrt{2} \approx 1.41$, $\sqrt{2} - 1 \approx 0.41$. The ball lands about 40% of the rock's radius beyond its base. With $g \approx 10$, and $\sqrt{g} \approx 3.2$, the maximal velocity is about $3.2\sqrt{R}$.

Double check: Things you can do: simply graph the two trajectories you came up with for a given value of R, and verify they do not intersect. Check the units of the final answer. Check that the ball lands beyond the base of the rock (it does).

Another way: Since the parabola has a maximal radius of curvature at at its apex, with a little geometrical reasoning you can prove that if the circle and parabola are tangent at the parabola's apex, and the parabola's radius of curvature there exceeds R, the two curves cannot intersect. It does work: calculate the parabola's radius of curvature, insist that it be larger than R, and the same condition results: $v \geqslant \sqrt{gR}$. I don't really have the stamina to work up a full geometric proof of that, however . . . perhaps one of you would do it for extra credit?

16. *HRW 4.53* A ball is thrown horizontally with speed v from the floor at the top of some stairs. The width and height of each step are both equal to l. **(a)** What should v be so that the ball barely clears the corner of the step that is N steps down? **(b)** How far along the next step (from its base) does the ball hit?

Solution: **Given:** The dimensions of a staircase, and the initial velocity of a ball which rolls off the staircase. Let the staircase width and height be d, and the ball's initial speed $|\vec{v}_i|$.

Find: Which step the ball first hits on its way down.

Sketch: We choose a cartesian coordinate system with x and y axes aligned with the stairs, as shown below. Let the origin be the point at which the ball leaves the topmost stair. The ball is launched horizontally off of the top step, and will follow a parabolic trajectory down the staircase. How do we determine which stair will first be hit? From the sketch, it is clear that we need to find at which point the ball's parabolic trajectory (solid curve) passes below a line connecting the right-most tips of each stair (dotted line).

Relevant equations: Based on our logic above, we need an equation for the ball's trajectory and an equation for the line describing the staircase boundary. The staircase itself is is composed of steps of equal height and width. Therefore, a line from the origin connecting the right-most tip of each stair (the dotted line in the figure) will have a slope of -1, and can be described by $y_s = -x$.

The ball will follow our now well-known parabolic trajectory. In this case, the launch angle is zero,

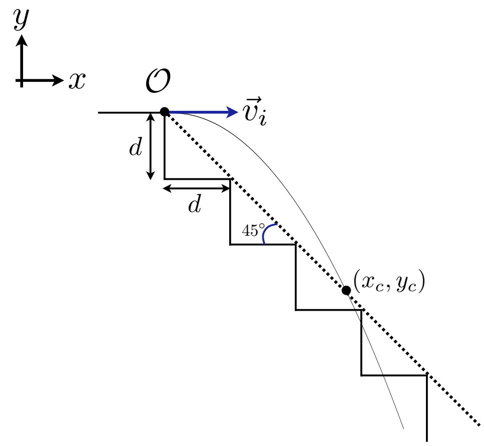


Figure 3: *A projectile launched horizontally off the top of a staircase.*

and the ball's motion is described by setting $\theta = 0$ in Eq. **??**:

$$y_b = -\frac{gx^2}{2|\vec{v}_i|^2} \tag{79}$$

We first need to find the x coordinate where $y_b = y_x$, which is the point where the parabolic trajectory dips below the line defining the staircase slope. Call this coordinate x_c. Given this coordinate, we need to determine how many stairs this distance corresponds to. The ratio of x_c to the stair width should give us this number. However, we must keep in mind the fact that the staircase is discrete: if we find that x_c corresponds to, for example, 4.7 stair widths, what does that mean? It means the ball crossed the fourth stair, but 70% of the way across the fifth one, its trajectory dipped below the line defining the staircase. Thus, the ball would hit the fifth stair.

What we need, then, is to *find the ratio of* x_c *and the stair width* d*, and take the next largest integer*. This gives us the number of the stair the ball first hits n_s. There is a mathematical function that does exactly what we want, for this operation, the *ceiling* function. It takes a real-valued argument x and gives back the next-highest integer. For example, if $x = 3.2$, then the ceiling of x is 4. The standard notation is $\lceil x \rceil = 4$.[v].

$$n_s = \left\lceil \frac{x_c}{d} \right\rceil$$

Symbolic solution: First, we need to find the point x_c where the ball's parabolic trajectory

[v]If you want to get all technical, $\lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, n \geqslant x\}$

intersects the staircase boundary line:

$$\begin{aligned}
y_b &= -\frac{gx^2}{2|v_i|^2} = y_s = -x \\
0 &= \frac{gx^2}{2|\vec{v}_i|^2} - x \\
0 &= x\left[\frac{gx}{2|\vec{v}_i|^2} + 1\right] \\
\implies \quad x_c &= \left\{0, \frac{2|\vec{v}_i|^2}{gd}\right\}
\end{aligned}$$

As usual, one of our answers is the trivial solution, the one where the ball never leaves the staircase ($x_c = 0$). The second solution is what we are after. The number of the stair that is first hit is then

$$n_s = \left\lceil \frac{x_c}{d} \right\rceil = \left\lceil \frac{2|\vec{v}_i|^2}{g} \right\rceil$$

Putting it another way, if you want to clear n_s stairs, then you must clear a horizontal distance of at least $(n_s - 1)d$ to make it to stair n_s. That implies

$$x_c = \frac{2|\vec{v}_i|^2}{gd} = (n_s - 1)\,d \qquad \implies \qquad |\vec{v}_i| = \frac{1}{2}gd\,(n_s - 1) \tag{80}$$

How far along the stair does it hit? Now you want to know when the y coordinate reaches the level of stair n_s, which is when $y = -dn_s$. Given $y(x) =! - dn_s$, solve for x, which will be the total horizontal distance covered when hitting stair n_s, then subtract off the distance of the previous stairs already cleared, dn_s.

17. *HRW 4.67* A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above ground level. The string breaks, and the stone flies off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone during the circular motion?

Solution: There are two parts to this. First, we have a projectile launched horizontally ($\theta = 0$) which goes horizontally some distance d and falls vertically some distance h. This will allow us to find the launch velocity. Second, the launch velocity is the same as the velocity of the stone in circular motion just before the string breaks. The velocity dictates the centripetal acceleration present during the circular motion, along with the radius of the circular path R.

First, the projectile motion. We will place the origin on the ground directly below the point at which the stone leaves the string. That makes the launch coordinates $(0, h)$ and the landing coordinates

$(d, 0)$. The trajectory of the stone is then

$$y(x) = h + x \tan\theta - \frac{gx^2}{2v_o^2 \cos^2\theta} = h - \frac{gx^2}{2v_o^2} \qquad \text{(note } \theta = 0\text{)} \tag{81}$$

Given $y(d) = 0$,

$$0 = h - \frac{gd^2}{2v_o^2} \tag{82}$$

$$v_o^2 = \frac{gd^2}{2h} \tag{83}$$

This velocity implies a centripetal acceleration during the whirling of

$$a_c = \frac{v_o^2}{R} = \frac{gd^2}{2Rh} \approx 470\,\text{m/s}^2 \tag{84}$$

That's about 47 g's for our little stone. No wonder the string broke.

18. *Morin PR 3.20* A typical front-loading washing machine might have a radius of 0.3 m and a spin cycle of 1000 revolutions per minute. What is the acceleration of a point on the surface of the drum at this spin rate? How many g's is this equivalent to?

Solution: Mostly just unit conversions here. Our angular velocity is 1000 rotations per minute, but we need that in rad/sec.

$$\omega = 1000\,\frac{\text{rev}}{\text{sec}}\,\frac{2\pi\,\text{rad}}{1\,\text{revolution}}\,\frac{1\,\text{min}}{60\,\text{sec}} \approx 105\,\text{rad/s} \tag{85}$$

Now keep in mind: the radian is not a real unit like meters or second, it is just a ratio of two lengths. We'll carry it through in our equations, just to make sure we haven't screwed anything up, but in the end we can drop it. Anyway: given the angular velocity, centripetal acceleration is easily found, remembering that $v = R\omega$.

$$a_c = \frac{v^2}{R} = R\omega^2 \approx 3.3 \times 10^3\,\text{m/s}^2 \approx 330g's \tag{86}$$